# PET5I103 ANALOG COMMUNICATION (3-0-2) (5th Sem ECE-ETC)

## MODULE-I

1. **SIGNALS AND SPECTRA:** An Overview of Electronic Communication Systems, Signal
and its Properties, Fourier series Expansion and its Use, The Fourier Transform, Orthogonal Representation of Signal.
2. **RANDOM VARIABLES AND PROCESSES:** Probability, Random variables, Useful Probability Density functions, Useful Properties and Certain Application Issues.
3. **AMPLITUDE MODULATION SYSTEMS:** Need for Frequency translation, Amplitude
Modulation (Double Side Band with Carrier DSB-C), Single Sideband Modulation (SSB) Other AM Techniques and Frequency Division Multiplexing.

## MODULE-II

4. **ANGLE MODULATION:** Angle Modulation, Tone Modulated FM Signal, Arbitrary Modulated FM signal, FM Modulators and Demodulators, Approximately Compatible SSB Systems.
5. **PULSE MODULATION AND DIGITAL TRANSMISSION OF ANALOG SIGNAL:**Analog
to Digital (Noisy Channel and Role of Repeater), Pulse Amplitude Modulation and Concept of Time division multiplexing, Digital Representation of Analog Signal

## MODULE-III

6. **MATHEMATICAL REPRESENTATION OF NOISE:** Some Sources of Noise, Frequency-domain Representation of Noise, Superposition of Noises, Linear Filtering of Noise.
7. **NOISE IN AMPLITUDE MODULATION SYSTEM:** Framework for Amplitude Demodulation, Single Sideband Suppressed Carrier (SSB-SC), Double Sideband Suppressed Carrier (DSB-SC), Double Sideband with Carrier (DSB-C).

## MODULE-IV

8. **NOISE IN FREQUENCY MODULATION SYSTEM:** An FM Receiving System, Calculation of Signal to Noise Ratio, Comparison of FM and AM, Pre emphasis and De-emphasis and SNR Improvement, Noise in Phase Modulation and Multiplexing Issues, The FM Demodulator using Feedback (FMFB).

## Additional Module (Terminal Examination-Internal)

1. AMPLITUDE MODULATION SYSTEMS:Radio Transmitter and Receiver.
2. PULSE MODULATION: Pulse Width Modulation and Pulse Position Modulation.
3. SYSTEM NOISE IN FREQUENCY MODULATION:Threshold in Frequency Modulation, Calculation of Threshold in an FM Discriminator.

# PEL6I101 COMMUNICATION ENGINEERING (6th Sem EEE)

## MODULE-I

INTRODUCTION: Elements of an Electrical Communication System, Communication Channels and their Characteristics, Mathematical Models for Communication Channels
FREQUENCY DOMAIN ANALYSIS OF SIGNALS AND SYSTEMS: Fourier series, Fourier Transforms, Power and Energy, Sampling and Band limited signals, Band pass

signals

**MODULE-II**

ANALOG SIGNAL TRANSMISSION AND RECEPTION: Introduction to modulation, Amplitude Modulation (AM), Angle Modulation, Radio and Television broadcasting

**MODULE-III**

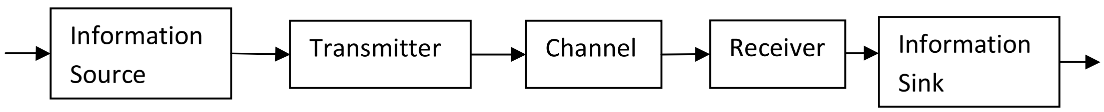
PULSE MODULATION SYSTEMS: Pulse amplitude modulation, Pulse Time Modulation
PULSE CODE MODULATION: PCM system, Intersymbol interference, Eye patterns, Equalization, Companding, Time Division Multiplexing of PCM signals, Line codes, Bandwidth of PCM system, Noise in PCM systems,

**MODULE-IV**

Delta Modulation (DM), Limitations of DM, Adaptive Delta Modulation, Noise in Delta Modulation, Comparison between
PCM and DM, Delta or Differential PCM (DPCM), S-Ary System

# MODULE-I

Elements of an Electrical Communication System




Speech, Music,

Image, Video, Instrumentation and measurement signals, EEG, ECG, Seismic, meteorological signals, Data etc

(Transducers required)

Analog

(Point-to-point, broadcast), Low Power, Medium Power, High Power, Digital

Additive,

Linear Time Invariant, Linear Time varying, Band limited, Power limited

Analog
Digital

Transducers

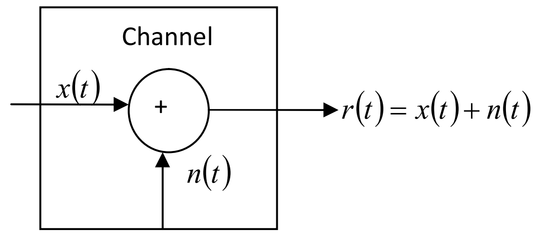
(Speaker, Picture tube etc)

Information is obtained from real life signals through the use of transducers. For example, speech is converted into a corresponding electrical signal by a microphone and moving picture signals are converted into the appropriate electrical signals by various cameras. The information so obtained is called a signal that becomes a function of time which is usually analog in nature. Signals may be described in time domain or in frequency domain. The frequency domain description of a signal is known as spectrum that would be covered subsequently. Data generated by the keystroke of a computer become the information when communication is made through e-mail.

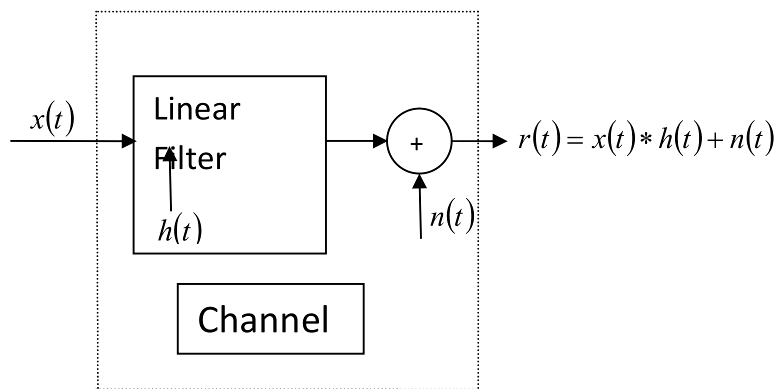
The transmitter may operate in a point-to-point mode or in a broadcast mode wherein there is a number of receivers corresponding to a single transmitter. It may be wired, wireless. The transmitter may also operate at different power levels depending upon the application, range of service and type of service. We get three distinct types of transmitters: simplex, half duplex and full duplex. The broadcast transmitters usually meant for entertainment purpose are simplex type as information flow is unidirectional. The receiver can not communicate back to the transmitter. In half duplex system, information can flow between the transmitter and the receiver in one direction only at a time, but not simultaneously. The walkie-talkie is an example of simplex type of communication. The telephone provides an example of a full duplex type of communication.

The channel may be modelled as

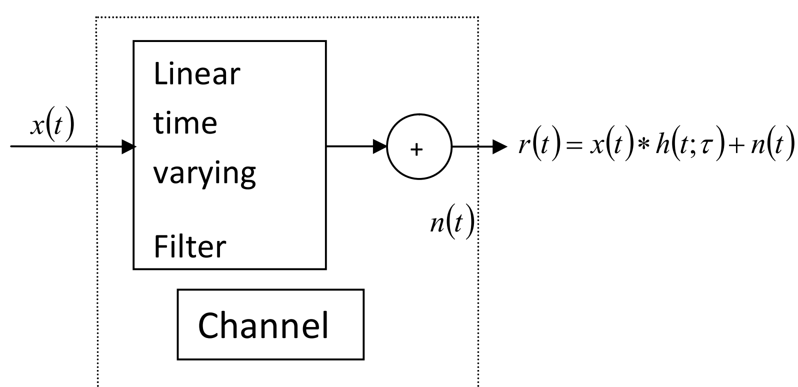
- Additive noise type: the channel introduces noise that is added to the transmitted signal (satellite channels)




- Linear time invariant (LTI) type: the channel behaves as a linear filter whose impulse response (or alternatively the transfer function) does not vary with respect to time. The transmitted signal is convolved with the impulse response to produce the channel output. (Leased land line telephone lines or simply the telephone channel)




- Linear time varying (LTV) type: The channel again, here behaves as a linear filter. However, unlike the LTI channel, the impulse response of the channel varies with respect to time. The channel output is observed to be a convolution of the transmitted signal and the time varying impulse response. Cellular channels provide a bright example of this kind of channel.



The receiver's function is to retrieve the original transmitted signal from noisy, distorted signals that arrive at its input. An analog receiver is entrusted with the task of replicating the original waveform from its noise corrupt and channel induced distorted versions. A digital receiver makes a decision (within a sampling interval) as to "which one out of M number of symbols".

Performance metric of receivers:

Signal to noise ratio (SNR) at receiver output fort the analog one
Probability of bit error or
Mean square error (MSE) for the digital type.

The sink is usually a speaker that reproduces speech signals from the corresponding electrical output or a picture tube that reproduces the picture. It may be a computer also that is intended to receive an e-mail.

**Electromagnetic Spectrum**

**Table No. 1.1 Allocation of frequencies for existing applications**

| S No | Frequency Range | Nomenclature | Application/Usage |
|---|---|---|---|
| 1 | 30 Hz- 300 Hz | Extremely low frequencies (ELF) | Underwater Communication |
| 2 | 300 Hz- 3. 0 KHz | Voice Frequency (VF) | Telephone |
| 3 | 3.0 KHz – 30 KHz | Very low frequencies (VLF) | Navigation |
| 4 | 30 KHz- 300 KHz | Low Frequency (LF) | Radio navigation |
| 5 | 300 KHz - 3 MHz | Medium Frequencies (MF) | AM radio broadcasting |
| 6 | 3 MHz- 30 MHz | High Frequencies (HF) | AM, Amateur radio, mobile |
| 7 | 30 MHz – 300 MHz | Very High Frequencies (VHF) | TV, FM, Mobile communications |
| 8 | 300 MHz- 3 GHz | Ultra High Frequencies (UHF) | TV, radar, satellite communications |
| 9 | 3 GHz- 30 GHz | Super High Frequencies (SHF) | Terrestrial microwave and satellite communications |
| 10 | $10^5$ GHz – $10^6$ GHz | Optical Frequencies | Optical communication |

**Signal Analysis: Fourier Series**

A signal is periodic if it repeats itself after a certain time; $x(t) = x(t+T)$ where $T$ is its period.

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \exp\left(\frac{j2\pi nt}{T}\right)$$

where $x_n = \frac{1}{T}\int_0^T x(t)\exp\left(-\frac{j2\pi nt}{T}\right)dt$

Two signals $x_1(t)$ and $x_2(t)$ are said to be orthogonal over a period T if their inner product is zero;

$\int_0^T x_1(t)x_2(t)dt = 0$ for the case when the signals are real valued functions.

For example: $V\sin 2\pi f_0 t$ and $V\cos 2\pi f_0 t$ , $V\sin 2\pi m f_0 t$ , $V\sin 2\pi n f_0 t$ are orthogonal to each other over the period $T$

The fundamental frequency is expressed as $f_0 = \frac{1}{T}$

- The signal must satisfy a set of conditions known as 'Dirichlet's conditions'
- These ae
- A) The signal is absolutely summable over its period

$$\int_0^T |x(t)|dt < \infty$$

The coefficient $a_n = \frac{2}{T}\int_0^T x(t)\cos\frac{2\pi nt}{T}dt$ and $b_n = \frac{2}{T}\int_0^T x(t)\sin\frac{2\pi nt}{T}dt$

$x_n = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$ and $\phi(x_n) = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$

$\frac{1}{n^\alpha}$ where $\alpha \geq 1$

Fourier transform of a function is evaluated as

$$X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)dt$$

**Q.1** Show that $\int_{-\infty}^{t} m(\lambda)d\lambda = m(t) * u(t)$

Proof:

$$m(t) * u(t) = \int_{-\infty}^{\infty} m(\lambda)u(t-\lambda)d\lambda = \int_{-\infty}^{t} m(\lambda)d\lambda$$

This is because $u(t-\lambda) = 0$ for $\lambda > t$ .

**Q.2**. Find the spectrum for a signal defined as

$$v(t) = \sin 2\pi f_1 t \cos 2\pi f_2 t$$

Soln: The signal is $v(t) = \sin 2\pi f_1 t \cos 2\pi f_2 t = \frac{1}{2}\{\sin[2\pi(f_1 + f_2)t] - \sin[2\pi(f_1 - f_2)t]\}$

The spectrum corresponding to the first term is

$$\frac{1}{2j}\{\delta[f + (f_1 + f_2)] - \delta[f - (f_1 + f_2)]\}$$

The spectrum corresponding to the first term is

$$\frac{1}{2j}\{\delta[f + (f_1 - f_2)] - \delta[f - (f_1 - f_2)]\}$$

**Q.3.** Find the spectrum of a signal defined as $t\frac{dg(t)}{dt}$

Soln: We know that

The function $g(t) \Leftrightarrow G(f)$, then $\frac{d}{dt}g(t) \Leftrightarrow j2\pi f G(f)$ and similarly,

$$\frac{d}{df}G(f) \Leftrightarrow (-j2\pi t)g(t)$$

Let us differentiate the function $g_1(t) = (-j2\pi t)g(t)$ once more with respect to time. Hence, we obtain

$$\frac{d}{dt}g_1(t) = \left[-j2\pi t\frac{d}{dt}g(t) - j2\pi g(t)\right] \Leftrightarrow j2\pi f\frac{d}{df}G(f)$$

From the linearity property of the Fourier transform operator, we have, corresponding to the second term of the expression,

$-j2\pi g(t) \Leftrightarrow -j2\pi G(f)$ (Multiplying $-j2\pi$ to both sides )

Thus, the Fourier transform of the function $t\frac{dg(t)}{dt}$ is obtained by writing

$$t\frac{dg(t)}{dt} \Leftrightarrow -\frac{1}{j2\pi}[j2\pi f G(f)] - \frac{1}{j2\pi}[j2\pi G(f)] = -[G(f) + fG(f)]$$

Q.4 Find the Fourier transform of a function given as $x(t) = (t-2)f(-2t)$

From the frequency differentiation property of the Fourier transform, we have

$$tx(t) \leftrightarrow \frac{1}{-j2\pi}\frac{d}{df}X(f)$$

$$\Rightarrow tx(-2t) \leftrightarrow \frac{1}{-j2\pi.2}\frac{d}{df}X\left(-\frac{f}{2}\right)$$

$$-2x(-2t) \leftrightarrow -2.\frac{1}{|-2|}X\left(-\frac{f}{2}\right) = -X\left(-\frac{f}{2}\right)$$

$$\Rightarrow (t-2)x(-2t) \leftrightarrow \frac{1}{-j2\pi.2}\frac{d}{df}X\left(-\frac{f}{2}\right) - X\left(-\frac{f}{2}\right)$$

**Q.4** Find the Fourier transform of a signal given as

$$y(t) = t\frac{dx}{dt}$$

Soln: Differentiating a function like $tx(t)$ in the time domain, we have

$$\frac{d}{dt}[tx(t)] = t\frac{d}{dt}x(t) + x(t)$$

$$\Rightarrow t\frac{d}{dt}x(t) = \frac{d}{dt}[tx(t)] - x(t)$$

From the above, we note that,

$$tx(t) \leftrightarrow \frac{1}{-j2\pi}\frac{d}{df}X(f)$$

$$\Rightarrow \frac{d}{dt}[tx(t)] \leftrightarrow j2\pi f.\frac{1}{-j2\pi}\frac{d}{df}X(f) = -f\frac{d}{df}X(f)$$

However, from the linearity principle, the time differentiated function has two parts; the transform corresponding to $y(t)$ and the other corresponding to $x(t)$. Therefore,

$$-f\frac{d}{df}X(f) = Y(f) + X(f)$$

$$\Rightarrow Y(f) = -\left[X(f) + f\frac{d}{df}X(f)\right]$$

$$\Rightarrow t\frac{dx}{dt} \Leftrightarrow -\left[X(f) + f\frac{d}{df}X(f)\right]$$

Q.5 Find the Fourier transform of a signal defined as

$$y(t) = x(1-t)$$

Soln: $y(t) = x(1-t) = x[-(t-1)]$

$$x(t) \leftrightarrow X(f)$$

We know that, for $x(-t) \leftrightarrow X^*(f)$

$$\Rightarrow x[-(t-1)] \leftrightarrow X^*(f)\exp(-j2\pi f)$$

This is because

$$\Im(x(-t)) = \int_{-\infty}^{\infty} x(-t)\exp(-j2\pi ft)dt$$

$$Let$$
$$-t = \lambda$$
$$\Rightarrow dt = d\lambda$$
$$\Rightarrow -\int_{\infty}^{-\infty} x(\lambda)\exp(j2\pi f\lambda)d\lambda$$
$$\Rightarrow \int_{-\infty}^{\infty} x(\lambda)\exp(j2\pi f\lambda)d\lambda$$
$$\Rightarrow \int_{-\infty}^{\infty} [x(\lambda)\exp(-j2\pi f\lambda)d\lambda]^*$$
$$= X^*(f)$$

Q.6 Find the Fourier transform of a signal given as

$$y(t) = (1-t)x(1-t)$$

Soln: As in the previous problem,

$$tx(1-t) \leftrightarrow \frac{1}{j2\pi}\frac{d}{df}Y(f) = \frac{1}{j2\pi}\frac{d}{df}\left[X^*(f)\exp(-j2\pi f)\right]$$
$$= \frac{1}{j2\pi}\left[-j2\pi X^*(f)\exp(-j2\pi f) + \frac{d}{df}X^*(f)\exp(-j2\pi f)\right]$$
$$= \left[\frac{1}{j2\pi}\frac{dX^*(f)}{df} - X^*(f)\right]\exp(-j2\pi f)$$

From the linearity property of the Fourier transform, we have

$$y(t) = (1-t)x(1-t) \leftrightarrow X^*(f)\exp(-j2\pi f) - \left[\frac{1}{j2\pi}\frac{dX^*(f)}{df} - X^*(f)\right]\exp(-j2\pi f)$$
$$= \left[2X^*(f) - \frac{1}{j2\pi}\frac{dX^*(f)}{df}\right]\exp(-j2\pi f)$$

The impulse has no mathematical or physical meaning unless it appears under the operation of integration. Two of the most significant integration properties are

I. Replication property

$$x(t)*\delta(t-t_0)=x(t-t_0)$$

To prove this, we write

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-t_0)d\tau$$

$$\begin{aligned}
&let\\
&t-\tau-t_0=\lambda\\
&\Rightarrow \tau=t-\lambda-t_0\\
&\Rightarrow d\tau=-d\lambda\\
&\Rightarrow \int_{\infty}^{-\infty} x(t-\lambda-t_0)\delta(\lambda)d\tau\\
&\Rightarrow \int_{\infty}^{\infty} x(t-\lambda-t_0)\delta(\lambda)d\tau
\end{aligned}$$

We know that, $\delta(\lambda)$ attains a value of 1 at $\lambda=0$. Therefore,

$$\begin{aligned}
&\int_{\infty}^{\infty} x(t-\lambda-t_0)\delta(\lambda)d\tau=x(t-t_0).1\\
&=x(t-t_0)
\end{aligned}$$

This is known as replication property.

II. Sampling property

We also have,

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt=x(t_0)$$

To prove this, we note that

$$\begin{aligned}
&t-t_0=\lambda\\
&\Rightarrow t=\lambda+t_0\\
&\Rightarrow dt=d\lambda
\end{aligned}$$

Hence the above integral becomes

$$\int_{-\infty}^{\infty} x(\lambda+t_0)\delta(\lambda)d\lambda$$

We know that, $\delta(\lambda)$ attains a value of 1 at $\lambda = 0$. Therefore, the above integral has just one value that is nonzero occurring at $\lambda = 0$ and this value is given as

$$x(t_0)$$

This completes the proof.

III.

Further, we have

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

This is because the impulse function has a value of 1 at $t = t_0$. Hence, only one value of the function $x(t)$ is retained which occurs at $t = t_0$.

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

This is because

Let us evaluate

$$\int_{-\infty}^{\infty} \delta(at)dt$$

$$at = x$$

Let $\Rightarrow t = \dfrac{x}{a}$

$$dt = \frac{dx}{a}$$

Therefore,

$$\int_{-\infty}^{\infty} \delta(at)dt = \frac{1}{a}\int_{-\infty}^{\infty} \delta(x)dx = \frac{1}{a}\delta(t)$$

IV. $x(t) * \delta(t-t_0) = x(t-t_0)$

To prove this, we write

$$x(t) * \delta(t-t_0)$$

$$= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-t_0)d\tau$$

Let

$$t - \tau - t_0 = \lambda$$
$$\Rightarrow t - \lambda - t_0 = \tau$$
$$\Rightarrow d\tau = -d\lambda$$

Substitution of the above in the integral gives us

$$\int_{-\infty}^{\infty} x(\tau)\delta(t - \tau - t_0)d\tau$$
$$= -\int_{\infty}^{-\infty} x(t - \lambda - t_0)\delta(\lambda)d\lambda$$
$$= x(t - t_0)$$

This is because the impulse function has a value of 1 at $t = t_0$. From the previous problem we get this.

V. $x(t - T_1) * \delta(t - T_2) = x(t - T_1 - T_2)$

$$x(t - T_1) * \delta(t - T_2)$$
$$= \int_{-\infty}^{\infty} x(\tau - T_1)\delta(t - \tau - T_2)d\tau$$

Let
$$t - \tau - T_2 = \lambda$$
$$\Rightarrow d\tau = -d\lambda$$
$$\Rightarrow \tau = t - \lambda - T_2$$

Thus,

$$\int_{-\infty}^{\infty} x(\tau - T_1)\delta(t - \tau - T_2)d\tau$$
$$= -\int_{\infty}^{-\infty} x(t - \lambda - T_1 - T_2)\delta(\lambda)d\lambda$$
$$= \int_{\infty}^{\infty} x(t - \lambda - T_1 - T_2)\delta(\lambda)d\lambda$$
$$= x(t - T_1 - T_2)$$

A little extension of the this result as applied to impulse functions give us

VI. $\delta(t - T_1) * \delta(t - T_2) = \delta(t - T_1 - T_2)$

Soln: Let us prove this using the Fourier transform properties.

We know that,

$$x(t) \leftrightarrow X(f)$$
$$\Rightarrow x(t-T_1) \leftrightarrow X(f)\exp(-j2\pi fT_1)$$
$$\Rightarrow \delta(t-T_1) \leftrightarrow \exp(-j2\pi fT_1)$$

Therefore,

$$\delta(t-T_1) * \delta(t-T_2) \leftrightarrow \exp(-j2\pi fT_1)\exp(-j2\pi fT_2)$$
$$= \exp[-j2\pi f(T_1+T_2)]$$
$$\leftrightarrow \delta(t-T_1-T_2)$$

Prove the duality theorem of Fourier transform which states that if $x(t) \leftrightarrow X(f)$, then $X(t) \leftrightarrow x(-f)$

Proof:

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df$$

Hence,

$$x(-t) = \int_{-\infty}^{\infty} X(f)\exp(-j2\pi ft)df$$

Let us interchange the roles of frequency and time in the above expression

Therefore,

$$x(-f) = \int_{-\infty}^{\infty} X(t)\exp(-j2\pi ft)dt = \Im(X(t))$$
$$\Rightarrow X(t) \leftrightarrow x(-f)$$

Find out the Fourier transform of $x(-t)$.

Soln: We know that,

$$x(t) = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df$$

Let $t = -\lambda$

Therefore,

$$x(-\lambda) = \int_{-\infty}^{\infty} X(f)\exp(-j2\pi\lambda f)df$$

Table 1.2 Some commonly used functions and their Fourier transforms

| | $x(t)$ | $X(f)$ |
|---|---|---|
| 1 | $rect\left(\frac{t}{T}\right)$ | $T$ sinc$(fT)$ |
| 2 | sinc$(2Wt)$ | $\frac{1}{2W} rect\left(\frac{f}{2W}\right)$ |
| 3 | $\exp(-at)u(t), a > 0$ | $\frac{1}{a + j2\pi f}$ |
| 4 | $\exp(-a\|t\|), a > 0$ | $\frac{2a}{a^2 + (2\pi f)^2}$ |
| 5 | $\exp(-\pi t^2)$ | $\exp(-\pi f^2)$ |
| 6 | $\begin{cases} 1 - \frac{\|t\|}{T}, \|t\| < T \\ 0, \|t\| \geq T \end{cases}$ | $T \sin c^2(fT)$ |
| 7 | $t\exp(-at)u(t)$ | $\frac{1}{(a + j2\pi f)^2}$ |
| 8 | sgn(t) | $\frac{1}{j\pi f}$ |
| 9 | $\|t\|$ | $\frac{-2}{(2\pi f)^2} = -\frac{1}{2\pi^2 f^2}$ |
| 10 | $u(t)$ | $\frac{1}{2}\left(\frac{1}{j\pi f} + \delta(f)\right)$ |
| 11 | $\exp(j2\pi f_c t)$ | $\delta(f - f_c)$ |
| 12 | $\sum_{n=-\infty}^{\infty} \delta(t - nT_0)$ | $\frac{1}{T_0} \sum_{m=\infty}^{\infty} \delta\left(f - \frac{m}{T_0}\right)$ |

Prove the 7th entry of Table 1.1 from the appropriate property of the Fourier transform.

Soln: The appropriate property that we use to prove this is the frequency domain differentiation which is

$$tx(t) \leftrightarrow \frac{1}{-j2\pi} \frac{d}{df} X(f)$$

As we note that,

$$\exp(-at)u(t), a > 0 \leftrightarrow \frac{1}{a + j2\pi f}$$

$$\Rightarrow t\exp(-at)u(t) \leftrightarrow -\frac{1}{j2\pi}\frac{d}{df}X(f)$$

$$= -\frac{1}{j2\pi}\frac{d}{df}\left(\frac{1}{a + j2\pi f}\right) = -\frac{1}{j2\pi}\frac{-j2\pi}{(a + j2\pi f)^2} = \frac{1}{(a + j2\pi f)^2}$$

This completes the proof.

Prove the 9th entry of Table 1.1 using appropriate properties of Fourier transform

Soln: We make use of the previous result. The function under consideration may be expressed as

$$|t| = \begin{cases} t & t \geq 0 \\ -t & t < 0 \end{cases}$$

The positive going part may also be considered as a limiting case of

$$\lim_{a \to 0} t\exp(-at)u(t)$$

Similarly, the negative going part may also be considered as the limiting case of the previous function however, with a reversed time

$$|t| = \begin{cases} t & t \geq 0 \\ -t & t < 0 \end{cases}$$
$$= \lim_{a \to 0}[t\exp(-at)u(t) - t\exp(at)u(-t)]$$

Let us combine the Fourier transforms of the two functions

$$X(f) = \frac{1}{(a + j2\pi f)^2} + \frac{1}{(a - j2\pi f)^2} = \frac{2[a^2 + (j2\pi f)^2]}{[a^2 - (j2\pi f)^2]}$$

For the limiting case of $a \to 0$, we have

$$\lim_{a \to 0} X(f) = \frac{2[a^2 + (j2\pi f)^2]}{[a^2 - (j2\pi f)^2]^2} = 2\frac{(j2\pi f)^2}{(j2\pi f)^4} = -\frac{2}{(2\pi f)^2}$$

Prove the 12th entry of Table 1.1.

Soln: The Fourier coefficient of this function is defined as

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\exp\left(-j2\pi\frac{n}{T_0}t\right)dt = \frac{1}{T_0}$$

Hence, the Fourier series of an impulse train is expressed as

$$x(t)=\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\exp\left(j2\pi\frac{nt}{T_0}\right)$$

The trigonometric Fourier series would consist of the coefficients $a_n$' s and $b_n$

$$a_n=\frac{2}{T_0}\int_0^{T_0}\delta(t)\cos\frac{2\pi nt}{T_0}dt=\frac{2}{T_0}$$

$$b_n=\frac{2}{T_0}\int_0^{T_0}\delta(t)\sin\frac{2\pi nt}{T_0}dt=0$$

From the 11th entry of this table, we note that

$$\exp\left(j2\pi\frac{n}{T_0}t\right)\leftrightarrow\delta\left(f-\frac{n}{T_0}\right)$$

Therefore, the Fourier series corresponding to an impulse train is expressed as

$$X(f)=\frac{1}{T_0}\sum_{m=-\infty}^{\infty}\delta\left(f-\frac{m}{T_0}\right)$$

The Fourier series of some commonly used waveforms

1. **Half wave rectified sine wave of amplitude $A$ volt**

$$x(t)=A\left(\frac{1}{\pi}+\frac{1}{2}\sin\omega t-\frac{2}{3\pi}\cos 2\omega t-\frac{2}{15\pi}\cos 4\omega t-\frac{2}{35\pi}\cos 6\omega t-\frac{2}{63}\cos 8\omega t\right)$$
$$=\left(\frac{A}{\pi}\right)+\frac{A}{2}\sin\omega t-\sum_{n=2}^{2m}\frac{2}{(n^2-1)\pi}\cos n\omega t$$

2. **Full wave rectified sine wave of peak amplitude $A$ volt**

$$x(t)=\frac{2A}{\pi}\left(1-\frac{2}{3\pi}\cos 2\omega t-\frac{2}{15\pi}\cos 4\omega t-\frac{2}{35\pi}\cos 6\omega t-\frac{2}{63}\cos 8\omega t\right)$$
$$=\left(\frac{2A}{\pi}\right)-\left(\frac{2A}{\pi}\right)\sum_{n=2}^{2m}\frac{2}{(n^2-1)}\cos n\omega t$$

3. **Rectangular or square wave of peak to peak amplitude $2A$ volt**

$$x(t)=\frac{4A}{\pi}\left(\sin\omega t+\frac{1}{3}\sin 3\omega t+\frac{1}{5}\sin 5\omega t+\frac{1}{7}\sin 7\omega t\right)$$
$$=\frac{4A}{\pi}\sum_{n=2k+1}\frac{1}{n}\sin n\omega t$$

4. **Triangular wave**

$$x(t)=\frac{8A}{\pi^2}\left(\sin\omega t-\frac{1}{9}\sin 3\omega t+\frac{1}{25}\sin 5\omega t-\frac{1}{49}\sin 7\omega t+...\right)$$

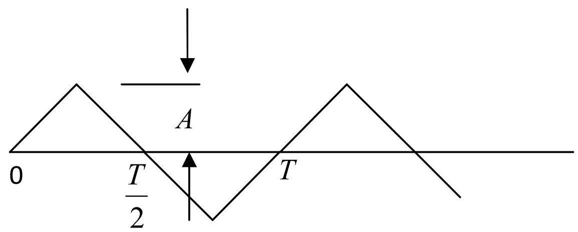



Fig.5

The waveform is expressed as

$$x(t)=\begin{cases}\frac{4A}{T}t & 0<t<\frac{T}{4}\\ -\frac{4A}{T}t+2A & \frac{T}{4}<t<\frac{3T}{4}\\ \frac{4A}{T}t-2A & \frac{3T}{4}<t<T\end{cases}$$

This function exhibits odd symmetry. Hence it contains the sine terms only. The coefficient $b_n$ is evaluated as

$$b_n=\frac{2}{T}\left[\int_0^{T/4}\frac{4A}{T}tdt+\int_{T/4}^{3T/4}\left(-\frac{4A}{T}t+2A\right)\sin n\omega_0 tdt\right]$$

$$=\frac{2}{T}\left[\frac{4A}{T}\int_0^{T/4}t\sin n\omega_0 tdt-\frac{4A}{T}\int_{T/4}^{3T/4}t\sin n\omega_0 tdt+2A\int_{T/4}^{3T/4}\sin n\omega_0 tdt\right]$$

Let us evaluate the above coefficient term by term. The first term gives us

$$\frac{2}{T}\left[\frac{4A}{T}\cdot\frac{1}{(n\omega_0)^2}\left|\{\sin n\omega_0 t-n\omega_0 t\cos n\omega_0 t\}\right|_0^{T/4}\right]$$

$$=\frac{8A}{(n\omega_0 T)^2}\left(\sin\frac{n\omega_0 T}{4}-\frac{n\omega_0 T}{4}\cos\frac{n\omega_0 T}{4}\right)$$

$$=\frac{8A}{4\pi^2 n^2}\left(\sin\frac{n\pi}{2}-\frac{n\pi}{2}\cos\frac{n\pi}{2}\right)$$

The second integral becomes

$$-\frac{2}{T}\frac{4A}{T}\int_{T/4}^{3T/4}t\sin n\omega_0 tdt$$

$$=-\frac{2}{T}\left[\frac{4A}{T}\cdot\frac{1}{(n\omega_0)^2}\left|\{\sin n\omega_0 t-n\omega_0 t\cos n\omega_0 t\}\right|_{T/4}^{3T/4}\right]$$

$$=-\frac{8A}{4n^2\pi^2}\left[\sin\frac{3n\omega_0 T}{4}-\frac{3n\omega_0 T}{4}\cos\frac{3n\omega_0 T}{4}-\sin\frac{n\omega_0 T}{4}+\frac{n\omega_0 T}{4}\cos\frac{n\omega_0 T}{4}\right]$$

$$=-\frac{8A}{4n^2\pi^2}\left[\sin\frac{3n\pi}{2}-\frac{3n\pi}{2}\cos\frac{3n\pi}{2}-\sin\frac{n\pi}{2}+\frac{n\pi}{2}\cos\frac{n\pi}{2}\right]$$

The third integral becomes

$$\frac{2}{T}2A\int_{T/4}^{3T/4}\sin n\omega_0 t\,dt$$

$$=\frac{4A}{T}\left(-\frac{1}{n\omega_0}\right)\left(\cos n\omega_0 t\right)\Big|_{T/4}^{3T/4}$$

$$=\frac{4A}{T}\left(-\frac{1}{n\omega_0}\right)\left(\cos\frac{3n\omega_0 T}{4}-\cos\frac{n\omega_0 T}{4}\right)$$

$$=\frac{4A}{T}\left(-\frac{1}{n\omega_0}\right)\left(\cos\frac{3n\pi}{2}-\cos\frac{n\pi}{2}\right)$$

The fourth integral is

$$\frac{2}{T}.\int_{3T/4}^{T}\left(\frac{4A}{T}t-2A\right)\sin n\omega_0 t dt$$

$$=\frac{2}{T}\left[\frac{4A}{T}.\frac{1}{(n\omega_0)^2}\left|\{\sin n\omega_0 t-n\omega_0 t\cos n\omega_0 t\}\right|_{3T/4}^{T}\right]$$

$$=\frac{8A}{(n\omega_0 T)^2}\left[\sin n\omega_0 T-n\omega_0 T\cos n\omega_0 T-\sin\frac{3n\omega_0 T}{4}+\frac{3n\omega_0 T}{4}\cos\frac{3n\omega_0 T}{4}\right]$$

$$=\frac{8A}{(2\pi n)^2}\left(\sin 2n\pi-2n\pi\cos 2n\pi-\sin\frac{3n\pi}{2}+\frac{3n\pi}{2}\cos\frac{3n\pi}{2}\right)$$

The fifth integral becomes

$$-2A.\frac{2}{T}\int_{3T/4}^{T}\sin n\omega_o t dt$$

$$=-\frac{4A}{T}\left(-\frac{1}{n\omega_0}\right)\left(\cos n\omega_0 T-\cos\frac{3n\omega_0 T}{4}\right)$$

Combining all the terms, we obtain

$$\frac{8A}{4\pi^2 n^2}\left(\sin\frac{n\pi}{2}-\frac{n\pi}{2}\cos\frac{n\pi}{2}\right)-\frac{8A}{4n^2\pi^2}\left[\sin\frac{3n\pi}{2}-\frac{3n\pi}{2}\cos\frac{3n\pi}{2}-\sin\frac{n\pi}{2}+\frac{n\pi}{2}\cos\frac{n\pi}{2}\right]$$

$$+\frac{4A}{T}\left(-\frac{1}{n\omega_0}\right)\left(\cos\frac{3n\pi}{2}-\cos\frac{n\pi}{2}\right)$$

$$+\frac{8A}{(n\omega_0 T)^2}\left(\sin 2n\pi-2n\pi\cos 2n\pi-\sin\frac{3n\pi}{2}+\frac{3n\pi}{2}\cos\frac{3n\pi}{2}\right)$$

$$+\frac{4A}{T}\left(\frac{1}{n\omega_0}\right)\left(\cos 2n\pi-\cos\frac{3n\pi}{2}\right)$$

For values of $n = 2m$ being even, the terms $\sin\frac{n\pi}{2}$ vanishes and $\cos\frac{n\pi}{2}$ becomes $(-1)^m$

The term $\cos\frac{3n\pi}{2} = 0$ always. Similarly, for odd values of $n = 2m+1$, the term

$$\cos\frac{(2m+1)\pi}{2} = 0$$

This is simplified to

$$\frac{8A}{4\pi^2 n^2}\left(\sin\frac{n\pi}{2}\right) - \frac{8A}{4n^2\pi^2}\left[\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2} + \frac{n\pi}{2}\cos\frac{n\pi}{2}\right]$$

$$+\frac{4A}{T}\left(-\frac{1}{n\omega_0}\right)\left(-\cos\frac{n\pi}{2}\right) + \frac{8A}{(n\omega_0 T)^2}\left(-2n\pi\cos 2n\pi - \sin\frac{3n\pi}{2}\right)$$

$$+\frac{4A}{T}\left(\frac{1}{n\omega_0}\right)(\cos 2n\pi)$$

$$= \frac{2A}{\pi^2 n^2}\left(\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2} - \sin\frac{3n\pi}{2}\right) - \frac{2A}{\pi^2 n^2}\left(\frac{n\pi}{2}\cos\frac{n\pi}{2}\right)$$

$$+\frac{4A}{2n\pi}\cos\frac{n\pi}{2} + \frac{8A}{4\pi^2 n^2}(-2n\pi) + \frac{4A}{2n\pi}$$

$$= \frac{2A}{\pi^2 n^2}\left(\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2} - in\frac{3n\pi}{2}\right)$$

$$= \frac{4A}{\pi^2 n^2}\left(\sin\frac{n\pi}{2} - \sin\frac{3n\pi}{2}\right)$$

$$= \frac{4A}{\pi^2 n^2}.2\left[\cos\left(\frac{n\pi + 3n\pi}{4}\right)\sin\left(\frac{n\pi - 3n\pi}{4}\right)\right]$$

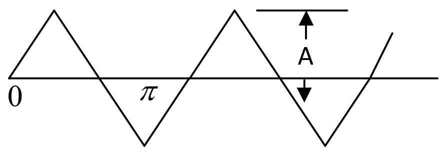
$$= \frac{8A}{\pi^2 n^2}.\left[-\cos n\pi \sin\frac{n\pi}{2}\right]$$

We note that, if $n = 1$, the above term is $\frac{8A}{\pi^2 n^2}$, for $n = 3$, it is $-\frac{8A}{\pi^2 n^2}$, for $n = 5$, the above term is $\frac{8A}{\pi^2 n^2}$. Hence the series amplitudes become alternately positive and negative and vary at the rate of $\frac{1}{n^2}$.

Thus, the Fourier series expansion of the triangular waveform as shown in Fig.5 is

$$x(t) = \frac{8A}{\pi^2}\left(\sin\omega_0 t - \frac{1}{9}\sin 3\omega_0 t + \frac{1}{25}\sin 5\omega_0 t - \frac{1}{49}\sin 7\omega_0 t + \ldots\right)$$

**6.** $x(t)=\frac{8A}{\pi^2}\left(\sin\omega t+\frac{1}{9}\sin 3\omega t+\frac{1}{25}\sin 5\omega t+\frac{1}{49}\sin 7\omega t+...\right)$



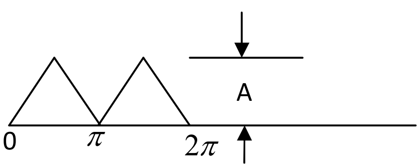

For the above triangular pulse, the Fourier series is obtained by noting that it can be obtained from Fig. 5 by shifting it by half a period.

$$x(t)=\frac{8A}{\pi^2}\left[\begin{array}{l}\sin\omega_0\left(t+\frac{T}{2}\right)-\frac{1}{9}\sin 3\omega_0\left(t+\frac{T}{2}\right)\\+\frac{1}{25}\sin 5\omega_0\left(t+\frac{T}{2}\right)-\frac{1}{49}\sin 7\omega_0\left(t+\frac{T}{2}\right)+...\end{array}\right]$$

$$=\frac{8A}{\pi^2}\left(\cos\omega_0 t+\frac{1}{9}\cos 3\omega_0 t+\frac{1}{25}\cos 5\omega_0 t+\frac{1}{49}\cos 7\omega_0 t+...\right)$$

**7. Full wave rectified triangular wave**

$$x(t)=\frac{A}{2}+\frac{4A}{\pi^2}\left(\sin\omega t-\frac{1}{9}\sin 3\omega t+\frac{1}{25}\sin 5\omega t-\frac{1}{49}\sin 7\omega t+...\right)$$




This waveform exhibits even symmetry. This has an average value given as

$$a_0=\frac{1}{T}\int_0^T x(t)dt$$

$$=\frac{1}{T}.\frac{1}{2}A.T=\frac{A}{2}$$

The waveform is expressed as

$$x(t)=\begin{cases}\frac{2A}{T}t & 0<t<\frac{T}{2}\\ -\frac{2A}{T}t+2A & \frac{T}{2}<t<T\end{cases}$$

The corresponding integrals become

$$\frac{2}{T}\left[\frac{2A}{T}.\frac{1}{(n\omega_0)^2}\left|\{\sin n\omega_0 t - n\omega_0 t\cos n\omega_0 t\}\right|_0^{T/2}\right]$$

$$=\frac{4A}{(n\omega_0 T)^2}\left(\sin\frac{n\omega_0 T}{2}-\frac{n\omega_0 T}{2}\cos\frac{n\omega_0 T}{2}\right)$$

$$=\frac{4A}{\pi^2 n^2}\left(\sin n\pi-\frac{n\pi}{2}\cos\frac{n\pi}{2}\right)$$

$$=-\frac{2A}{n\pi}\cos\frac{n\pi}{2}$$

The second integral becomes

$$-\frac{2}{T}\frac{2A}{T}\int_{T/2}^{T} t\sin n\omega_0 t dt$$

$$=-\frac{2}{T}\left[\frac{2A}{T}.\frac{1}{(n\omega_0)^2}\left|\{\sin n\omega_0 t - n\omega_0 t\cos n\omega_0 t\}\right|_{T/2}^{T}\right]$$

$$=-\frac{4A}{n^2\pi^2}\left[\sin n\omega_0 T - n\omega_0 T\cos n\omega_0 T - \sin\frac{n\omega_0 T}{2}+\frac{n\omega_0 T}{2}\cos\frac{n\omega_0 T}{2}\right]$$

$$=-\frac{4A}{n^2\pi^2}\left[\sin n\pi - n\pi\cos n\pi - \sin\frac{n\pi}{2}+\frac{n\pi}{2}\cos\frac{n\pi}{2}\right]$$

$$=-\frac{4A}{n^2\pi^2}\left[-n\pi\cos n\pi - \sin\frac{n\pi}{2}+\frac{n\pi}{2}\cos\frac{n\pi}{2}\right]$$

$$=\frac{4A}{n\pi}\cos n\pi+\frac{4A}{n^2\pi^2}\sin\frac{n\pi}{2}-\frac{2A}{n\pi}\cos\frac{n\pi}{2}$$

The third integral becomes

$$\frac{2}{T}2A\int_{T/2}^{T}\sin n\omega_0 t dt$$

$$=\frac{4A}{T}\left(-\frac{1}{n\omega_0}\right)\left(\cos n\omega_0 t\right)\Big|_{T/2}^{T}$$

$$=\frac{4A}{T}\left(-\frac{1}{n\omega_0}\right)\left(\cos n\omega_0 T-\cos\frac{n\omega_0 T}{2}\right)$$

$$=\frac{4A}{T}\left(-\frac{1}{n\omega_0}\right)\left(\cos n\pi-\cos\frac{n\pi}{2}\right)$$

$$-\frac{2A}{n\pi}\cos\frac{n\pi}{2}+\frac{4A}{n\pi}\cos n\pi+\frac{4A}{n^2\pi^2}\sin\frac{n\pi}{2}-\frac{2A}{n\pi}\cos\frac{n\pi}{2}$$
$$+\frac{4A}{T}\left(-\frac{1}{n\omega_0}\right)\left(\cos n\pi-\cos\frac{n\pi}{2}\right)$$
$$=\frac{4A}{n^2\pi^2}\sin\frac{n\pi}{2}-\frac{4A}{n\pi}\cos\frac{n\pi}{2}+\frac{4A}{n\pi}\cos n\pi+\frac{4A}{n\pi}\cos\frac{n\pi}{2}-+\frac{4A}{n\pi}\cos n\pi$$
$$=\frac{4A}{n^2\pi^2}\sin\frac{n\pi}{2}$$
$$=\frac{4A}{n^2\pi^2}(-1)^n$$

Hence, the Fourier series becomes

$$x(t)=\frac{A}{2}+\frac{4A}{\pi^2}\sum_{n=2m+1}\frac{(-1)^n}{n^2}\sin n\omega_0 t$$

**8. Trapezoidal waveform**

The waveform is expressed as

$$x(t)=\begin{cases}\frac{6A}{T}t & 0<t<\frac{T}{6}\\ A & \frac{T}{6}<t<\frac{T}{3}\\ -\frac{6A}{T}t+3A & \frac{T}{3}<t<\frac{2T}{3}\\ -A & \frac{2T}{3}<t<\frac{5T}{6}\\ \frac{6A}{T}t-6A & \frac{5T}{6}<t<T\end{cases}$$

The first integral is

$$\frac{2}{T}.\frac{6A}{T}\int_0^{\frac{T}{6}} t\sin n\omega_0 t dt$$
$$=\frac{12A}{(n\omega_0 T)^2}\left(\sin n\omega_0 t-n\omega_0 t\cos n\omega_0 t\right)\Big|_0^{T/6}$$
$$=\frac{12A}{4\pi^2 n^2}\left(\sin\frac{n\omega_0 T}{6}-\frac{n\omega_0 T}{6}\cos\frac{n\omega_0 T}{6}\right)$$
$$=\frac{3A}{\pi^2 n^2}\left(\sin\frac{n\pi}{3}-\frac{n\pi}{3}\cos\frac{n\pi}{3}\right)$$

The second integral is

$$\frac{2}{T}.A\int_{T/6}^{T/3}\sin n\omega_0 t dt$$

$$= -\frac{2A}{n\omega_0 T}\cos n\omega_0 t\Big|_{T/6}^{T/3}$$

$$= -\frac{2A}{n2\pi}\left(\cos\frac{n\omega_0 T}{3} - \cos\frac{n\omega_0 T}{6}\right)$$

$$= -\frac{A}{n\pi}\left(\cos\frac{2n\pi}{3} - \cos\frac{n\pi}{3}\right)$$

The third integral is

$$\frac{2}{T}\int_{T/3}^{2T/3}\left(-\frac{6A}{T}t + 3A\right)\sin n\omega_0 t dt$$

$$= -\frac{3}{\pi^2 n^2}\left(\sin\frac{4\pi n}{3} - \frac{4\pi n}{3}\cos\frac{4\pi n}{3} - \sin\frac{2\pi n}{3} + \frac{2\pi n}{3}\cos\frac{2\pi n}{3}\right)$$

$$-\frac{3A}{\pi n}\left(\cos\frac{4\pi n}{3} - \cos\frac{2\pi n}{3}\right)$$

We note that, the term

$$-\frac{3A}{\pi n}\left(\cos\frac{4\pi n}{3} - \cos\frac{2\pi n}{3}\right)$$

$$= -\frac{3A}{\pi n}.2\sin\left(\frac{4\pi n + 2\pi n}{3.2}\right)\cos\left(\frac{2\pi n - 4\pi n}{2.3}\right) = 0$$

The fourth integral is

$$-\frac{2}{T}.A\int_{2T/3}^{5T/6}\sin n\omega_0 t dt$$

$$= \frac{A}{n\pi}\left(\cos\frac{5n\pi}{3} - \cos\frac{4n\pi}{3}\right)$$

The fifth integral is

$$\frac{2}{T}\int_{5T/6}^{6T/6}\left(\frac{6A}{T}t - 6A\right)\sin n\omega_0 t dt$$

$$= \frac{3}{\pi^2 n^2}\left(\sin 2\pi n - 2\pi n\cos 2\pi n - \sin\frac{5\pi n}{3} + \frac{5\pi n}{3}\cos\frac{5\pi n}{3}\right)$$

$$+\frac{6A}{\pi n}\left(\cos 2n\pi - \cos\frac{5\pi n}{3}\right)$$

Combining all the terms, we have

$$\frac{3A}{\pi^2 n^2}\left(\sin\frac{n\pi}{3}-\frac{n\pi}{3}\cos\frac{n\pi}{3}\right)-\frac{A}{n\pi}\left(\cos\frac{2n\pi}{3}-\cos\frac{n\pi}{3}\right)$$
$$-\frac{3A}{\pi^2 n^2}\left(\sin\frac{4\pi n}{3}-\frac{4\pi n}{3}\cos\frac{4\pi n}{3}-\sin\frac{2\pi n}{3}+\frac{2\pi n}{3}\cos\frac{2\pi n}{3}\right)$$
$$+\frac{A}{n\pi}\left(\cos\frac{5n\pi}{3}-\cos\frac{4n\pi}{3}\right)$$
$$+\frac{3A}{\pi^2 n^2}\left(\sin 2\pi n-2\pi n\cos 2\pi n-\sin\frac{5\pi n}{3}+\frac{5\pi n}{3}\cos\frac{5\pi n}{3}\right)$$
$$+\frac{6A}{\pi n}\left(\cos 2n\pi-\cos\frac{5\pi n}{3}\right)$$
$$=-\frac{A}{n\pi}\cos\frac{2n\pi}{3}+\frac{4A}{\pi n}\cos\frac{4\pi n}{3}-\frac{2A}{\pi n}\cos\frac{2\pi n}{3}+\frac{A}{n\pi}\left(\cos\frac{5n\pi}{3}-\cos\frac{4n\pi}{3}\right)$$
$$-\frac{6A}{\pi n}-\frac{5A}{\pi n}\cos\frac{5\pi n}{3}+\frac{6A}{\pi n}-\frac{6A}{\pi n}\cos\frac{5\pi n}{3}$$
$$=\frac{3A}{n\pi}\left(\cos\frac{4\pi n}{3}-\cos\frac{2\pi n}{3}\right)$$
$$=\frac{3A}{n\pi}\left(\sin\frac{4\pi n+2\pi n}{2.3}.\sin\frac{4\pi n-2\pi n}{2.3}\right)=0$$
$$=\frac{3A}{\pi^2 n^2}\left(\sin\frac{n\pi}{3}-\sin\frac{4\pi n}{3}+\sin\frac{2\pi n}{3}-\sin\frac{5\pi n}{3}\right)\textit{Combining all } \sin\theta \textit{ terms}$$
$$=\frac{3A}{\pi^2 n^2}\left(\sin\frac{n\pi}{3}+\sin\frac{n\pi}{3}+\sin\frac{2\pi n}{3}+\sin\frac{2\pi n}{3}\right)\quad \Theta\ \sin(\pi+\theta)=-\sin\theta$$
$$=\frac{3.2A}{\pi^2 n^2}\left(\sin\frac{n\pi}{3}+\sin\frac{2n\pi}{3}\right)=\frac{3.2.2A}{\pi^2 n^2}\sin\left(\frac{n\pi+2n\pi}{3.2}\right)\cos\left(\frac{n\pi-2n\pi}{3.2}\right)$$
$$=\frac{12A}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{6}\right)\quad \textit{Here n can not be even,}\quad \textit{it can not be a}$$
*multiple of* 3 *either. The allowed values of n are* 1, 5, 7, 11...
$$=\frac{12A}{\pi^2 n^2}(\pm 1)\frac{\sqrt{3}}{2}=\frac{6\sqrt{3}}{\pi^2 n^2}$$
If we combine all the cosine terms, the result is zero.
The desired Fourier series is
$$x(t)=\frac{6\sqrt{3}A}{\pi^2}\left(\sin\omega_0 t-\frac{1}{25}\sin 5\omega_0 t+\frac{1}{49}\sin 7\omega_0 t-\right)$$

**9. A periodic impulse sequence (Impulse train)**

$$\delta_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

where $T$ is its period.

The coefficient $a_0$ is evaluated as

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)dt = \frac{1}{T}$$

Similarly,

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2}\delta(t)\cos n\frac{2\pi}{T}t\,dt = \frac{2}{T}$$

As the delta function train is an even function of time, the coefficient $b_n$ is zero.

The Fourier series corresponding to such an impulse train is expressed as

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} x_n \exp\left(jn\frac{2\pi}{T}t\right) = \frac{1}{T}\sum_{n=-\infty}^{\infty}\exp\left(jn\frac{2\pi}{T}t\right)$$

This is because the coefficient $c_n = \frac{1}{T}$ for the delta train

The generalized Fourier series of any arbitrary periodic signal is expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \exp\left(\frac{j2\pi}{T}nt\right)$$

$$\Rightarrow \Im[x(t)] = \Im\left[\sum_{n=-\infty}^{\infty} x_n \exp\left(\frac{j2\pi}{T}nt\right)\right]$$

$$= \sum_{n=-\infty}^{\infty} x_n \Im\left[\exp\left(\frac{j2\pi}{T}nt\right)\right]$$

$$= \sum_{n=-\infty}^{\infty} x_n \Im\left[\exp\left(\frac{j2\pi}{T}nt\right)\right]$$

$$= \sum_{n=-\infty}^{\infty} x_n \delta\left(f - \frac{n}{T}\right)$$

Here, we define just one period of the wave given as

$$x_T(T) = \begin{cases} x(t) & -T/2 \le t \le T/2 \\ 0 & \text{otherwise} \end{cases}$$

Any arbitrary periodic signal may be expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} x(t - nT) = x_T(t) * \sum_{n=-\infty}^{\infty}\delta(t - nT)$$

Taking the Fourier transform of both the sides, we have

$$\Im[x(t)] = \Im\left[\sum_{n=-\infty}^{\infty} x(t-nT)\right] = \Im\left[x_T(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT)\right]$$
$$= X_T(f)\frac{1}{T}\sum_{m=-\infty}^{\infty} \delta\left(f-\frac{m}{T}\right) = \frac{1}{T}\sum_{m=-\infty}^{\infty} X_T\left(\frac{m}{T}\right)\left(f-\frac{m}{T}\right) = X(f)$$

Comparing the two results, we have

$$x_n = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_T\left(\frac{n}{T}\right)$$

Hence, the generalized Fourier series of any arbitrary periodic signal is expressed as, using this result

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \exp\left(\frac{j2\pi}{T}nt\right)$$
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_T\left(\frac{n}{T}\right)\exp\left(\frac{j2\pi}{T}nt\right)$$

Taking the Fourier transform of both the sides, we have

$$\Im[x(t)] = \Im\left[\sum_{n=-\infty}^{\infty} x(t-nT)\right] = \Im\left[x_T(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT)\right]$$
$$= X_T(f)\frac{1}{T}\sum_{m=-\infty}^{\infty} \delta\left(f-\frac{m}{T}\right) = \frac{1}{T}\sum_{m=-\infty}^{\infty} X_T\left(\frac{m}{T}\right)\left(f-\frac{m}{T}\right) = X(f)$$

For the sampled signal case, when $t = nT$, we have

$$x(nT) = x_T(nT) * \delta(nT)$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(t-nT) = x_T(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

Taking the Fourier transform of both the sides,

$$\Im[x(t)] = \Im\left[\sum_{n=-\infty}^{\infty} x(t-nT)\right] = \Im\left[x_T(t) * \sum_{n=-\infty}^{\infty} \delta(t-nT)\right]$$
$$= X(f) = \sum_{n=-\infty}^{\infty} \Im[x(t-nT)] = \sum_{n=-\infty}^{\infty} X_T(f)\exp(j2\pi fnT)$$

We note that,

$$\delta(t-mT) \leftrightarrow \exp(-j2\pi fmT)$$

Hence, the Fourier transform of $\delta_T(t) = \sum_{m=-\infty}^{\infty} \delta(t-mT) \leftrightarrow \sum_{m=-\infty}^{\infty} \exp(-j2\pi fmT)$ is

Q. Find the spectrum of a full wave rectified sine wave from fundamentals.

A sampled signal is expressed as

$$x_\delta(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s) = \sum_{n=-\infty}^{\infty}x(nT_s)\delta(t - nT_s)$$

$$\Rightarrow \Im[x_\delta(t)] = \Im\left[x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s)\right] = X(f) * \Im\left[\sum_{n=-\infty}^{\infty}\delta(t - nT_s)\right]$$

$$= X(f)\frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta\left(f - \frac{n}{T_s}\right)$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}X\left(\frac{n}{T_s}\right)\delta\left(f - \frac{n}{T_s}\right)$$

$$= \Im\left[\sum_{n=-\infty}^{\infty}x(nT_s)\delta(t - nT_s)\right]$$

$$= \sum_{n=-\infty}^{\infty}x(nT_s)\Im[\delta(t - nT_s)]$$

$$= \sum_{n=-\infty}^{\infty}x(nT_s)\exp\left(-\frac{j2\pi nt}{T_s}\right)$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}X\left(f - \frac{n}{T_s}\right)$$

If we compare both the sides of the transform, we note that,

$$\sum_{n=-\infty}^{\infty}x(nT_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty}X\left(\frac{n}{T_s}\right)$$

Find out the Fourier transform of a Gaussian pulse given as

$$x(t) = \exp(-\pi t^2)$$

Soln: The Fourier transform is expressed as

$$X(f) = \int_{-\infty}^{\infty}\exp(-\pi t^2)\exp(-j2\pi ft)dt$$

$$= \int_{-\infty}^{\infty}\exp-\left(\pi t^2 + j2\pi ft\right)dt$$

$$= \int_{-\infty}^{\infty}\exp\left[-\left(\pi t^2 + j2\pi ft + \pi f^2 - \pi f^2\right)\right]dt$$

By adding and subtracting a term like $\pi f^2$ to the argument of the exponential function

$$X(f)=\int_{-\infty}^{\infty}\exp\left[-\left(\pi t^2+j2\pi ft+\pi f^2-\pi f^2\right)\right]dt$$

$$=\exp\left(-\pi f^2\right)\int_{-\infty}^{\infty}\exp\left[-\sqrt{\pi}\left(t+jf\right)^2\right]dt$$

Let $\sqrt{\pi}\left(t+jf\right)=u$

$\Rightarrow\sqrt{\pi}\,dt=du$

Substituting this in the above, we get

$$X(f)=\frac{2}{\sqrt{\pi}}\exp\left(-\pi f^2\right)\int_{0}^{\infty}\exp\left[-u^2\right]du$$

$$=\frac{2}{\sqrt{\pi}}\exp\left(-\pi f^2\right)\frac{\sqrt{\pi}}{2}$$

$$=\exp\left(-\pi f^2\right)$$

Inference: The spectrum of a Gaussian pulse is also another Gaussian pulse

Second Method:

This can also be derived by another method.

Frequency domain differentiation of a given signal gives us

$$x(t)\Leftrightarrow X(f)$$

$$\Rightarrow\frac{dX(f)}{df}\Leftrightarrow -j2\pi t x(t)$$

Suppose, we have a signal that is described by a first order differential equation expressed as

$$\frac{dx(t)}{dt}=-2\pi t x(t)$$

Taking the transform of both the sides, we have

$$j2\pi f X(f)=-j\frac{dX(f)}{df}$$

$$\Rightarrow X(f)=\exp\left(-\pi f^2\right)$$

If $x(t)$ is a continuous signal bandlimited to $\omega_m$ radians per second, then show that

$\frac{k}{\pi}\left[x(t)*\sin c(kt)\right]=x(t)$ for $k\geq\omega_m$

Proof:

$\frac{k}{\pi}[x(t) * \sin c(kt)]$ becomes in the frequency domain,

$$\frac{k}{\pi}\left[X(f).\frac{\pi}{k}\Pi\left(\frac{f}{2k}\right)\right] = X(f) \quad -k \le \omega \le k$$

Taking the inverse transform we note that, in the range of $-k \le \omega \le k$, the signal would be exactly equal to $x(t)$ for a frequency range of $k \ge \omega_m$.

We note that, in order to replicate the function $x(t)$, the condition is that $k \ge \omega_m$ otherwise for $k < \omega_m$, multiplication of the two functions in the frequency domain would result in spectrum mutilation of $X(f)$

Hence show that,

$\frac{k}{\pi}[\sin c(\omega_m t) * \sin c(\omega_m t)] = \sin c(\omega_m t)$ for $\omega_n \ge \omega_m$

Proof:

Use of the above result gives us

$$\frac{\omega_n}{\pi}\left[\frac{\pi}{\omega_m}\Pi\left(\frac{f}{2\omega_m}\right).\frac{\pi}{\omega_n}\Pi\left(\frac{f}{2\omega_n}\right)\right] = \frac{\pi}{\omega_m}\Pi\left(\frac{f}{2\omega_m}\right) \quad \text{for } \omega_n \ge \omega_m$$

Taking the inverse transform of the above result we get

$$\frac{\omega_n}{\pi}\left[\frac{\pi}{\omega_m}\Pi\left(\frac{f}{2\omega_m}\right).\frac{\pi}{\omega_n}\Pi\left(\frac{f}{2\omega_n}\right)\right] = \frac{\pi}{\omega_m}\Pi\left(\frac{f}{2\omega_m}\right) \Leftrightarrow \sin c(\omega_m t) \text{ for } \omega_n \ge \omega_m$$

| S No. | Nomenclature | Mathematical Description |
|---|---|---|
| 1 | Linearity | $ax_1(t)+bx_2(t) \Leftrightarrow aX_1(f)+bX_2(f)$ |
| 2 | Time Scaling | $x(at) \Leftrightarrow \frac{1}{\lvert a \rvert}X\left(\frac{f}{a}\right)$ |
| 3 | Duality | If $x(t) \Leftrightarrow X(f)$ *then* $X(t) \Leftrightarrow x(-f)$ |
| 4 | Time Shifting | $x(t-t_0) \Leftrightarrow X(f)\exp(-j2\pi f t_0)$ |
| 5 | Frequency Shifting (Modulation Theorem) | $x(t)\exp(j2\pi f_c t) \Leftrightarrow X(f-f_c)$ |
| 6 | Area under $x(t)$ | $\int_{-\infty}^{\infty} x(t)dt = X(0)$ |

| | | |
|---|---|---|
| 7 | Area under $X(f)$ | $\int_{-\infty}^{\infty} X(f)df = x(0)$ |
| 8 | Time domain differentiation | $\frac{dx(t)}{dt} \Leftrightarrow j2\pi f X(f)$ |
| 9 | Time domain integration | $\int_{-\infty}^{t} x(\lambda)d\lambda \Leftrightarrow \frac{1}{j2\pi f} X(f) + \frac{X(0)}{2}\delta(f)$ |
| 10 | Frequency domain differentiation | $\frac{dX(f)}{df} \Leftrightarrow -j2\pi t x(t)$ |
| 11 | Complex conjugation | $x^*(t) \Leftrightarrow X^*(-f)$ |
| 12 | Re $x(t)$ | $\frac{1}{2}\left[X(f) + X^*(-f)\right]$ |
| 13 | Im $x(t)$ | $\frac{1}{2j}\left[X(f) - X^*(-f)\right]$ |
| 14 | Multiplication in time domain | $x_1(t)x_2(t) \Leftrightarrow \int_{-\infty}^{\infty} X_1(\lambda)X_2(f-\lambda)d\lambda$ |
| 15 | Time domain convolution | $\int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)d\tau \Leftrightarrow X_1(f)X_2(f)$ |
| 16 | Parseval's Theorem | $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$ |
| 17 | Rayleigh's Theorem | $\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \int_{-\infty}^{\infty} \lvert x(f)\rvert^2 df$ |
| 18 | Moments Property | $\int_{-\infty}^{\infty} t^n x(t)dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)\Big\rvert_{f=0}$ |

Prove property 16 in table 1.3

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$$

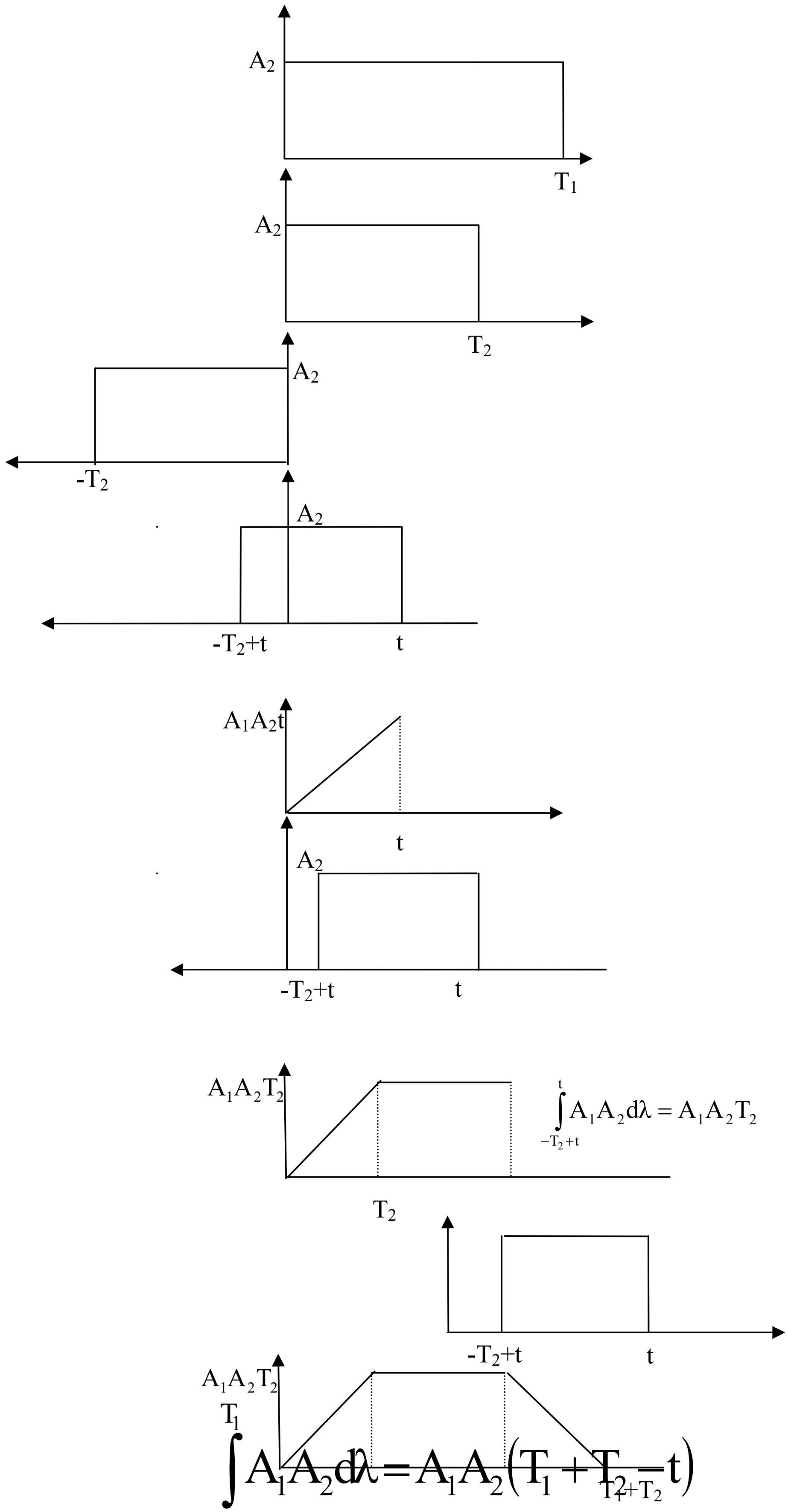
Proof:

$$\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)\exp(j2\pi ft)df y(t)dt$$

$$= \int_{-\infty}^{\infty} X(f)df\left[y(t)\exp(j2\pi ft)dt\right] = \int_{-\infty}^{\infty} X(f)df\left[y(t)\{\exp(j2\pi ft)\}^* dt\right]$$

$$= \int_{-\infty}^{\infty} X(f)dfY^*(f)$$

Next we show the convolution of two rectangular pulses of different amplitudes and different durations. The result is observed to be a trapezoidal pulse having a duration equal to the sum of the durations of the individual pulses.




$$\int_{-T_2+t}^{t} A_1A_2\,d\lambda = A_1A_2T_2$$

$$\int^{T_1} A_1A_2\,d\lambda = A_1A_2(T_1+T_2-t)$$

**Objective: Fourier Transform of Periodic Signals**

We have a periodic signal $x(t) = x(t + T_0)$ having a period $T_0$ that satisfies the Dirichlet's conditions. As we have seen previously, this signal is expressed as a linear weighted combinations of its Fourier series coefficients $\{x_n\}$ as

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \exp\left(\frac{j2\pi n}{T_0}t\right)$$

Taking the Fourier transform of both the sides we get

$$X(f) = \Im(x(t)) = \Im\left(\sum_{n=-\infty}^{\infty} x_n \exp\left(\frac{j2\pi n}{T_0}t\right)\right)$$

$$= \sum_{n=-\infty}^{\infty} x_n \Im\left[\exp\left(\frac{j2\pi n}{T_0}t\right)\right] = \sum_{n=-\infty}^{\infty} x_n \delta\left(f - \frac{n}{T_0}\right)$$

We observe the following from the above:

- That Fourier transform of a periodic signal $x(t)$ consists of a sequence of impulses in frequency at multiples of the fundamental frequency of the periodic signal.
- The weights of the impulses are just the Fourier series coefficients of the periodic signal
- Thus we obtain a discrete or line spectrum corresponding to a periodic signal
- Properties of Fourier transform would be utilized to compute the Fourier series coefficients as follows

We define a truncated signal $x_{T_0}(t)$ as

$$x_{T_0}(t) = \begin{cases} x(t) & -\frac{T_0}{2} < t < \frac{T_0}{2} \\ 0 & \text{otherwise} \end{cases}$$

This only means that we are just considering one period of the signal and we have set other periods to zero. The periodic signal is restored by repeating this truncated signal with a period of $T_0$. Hence, we get back our signal as

$$x(t) = \sum_{n=-\infty}^{\infty} x_{T_0}(t - nT_0) = x(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

Now, apply Fourier transform to both the sides.

$$X(f)=\Im[x(t)]=\Im\sum_{n=-\infty}^{\infty}x_{T_0}(t-nT_0)=\Im\left[x(t)*\sum_{n=-\infty}^{\infty}\delta(t-nT_0)\right]$$

$$=X_{T_0}(f)\frac{1}{T_0}\sum_{m=-\infty}^{\infty}\delta\left(f-\frac{m}{T_0}\right)$$

$$=\frac{1}{T_0}\sum_{m=-\infty}^{\infty}X_{T_0}(f)\delta\left(f-\frac{m}{T_0}\right)$$

Let us compare both the transforms. We can immediately see that

$$X(f)=\sum_{n=-\infty}^{\infty}x_n\delta\left(f-\frac{n}{T_0}\right)=\frac{1}{T_0}\sum_{m=-\infty}^{\infty}X_{T_0}\left(\frac{m}{T_0}\right)\delta\left(f-\frac{m}{T_0}\right)$$

$$x_n=\frac{1}{T_0}X_{T_0}\left(\frac{n}{T_0}\right)$$

The following steps are followed to find out the Fourier series coefficients $\{x_n\}$:

- Truncate the signal to just one period.
- Determine the Fourier transform of this truncated signal.
- Evaluate the Fourier transform of the truncated signal at a frequency $f=\frac{n}{T_0}$ to obtain the $n-$th harmonic and multiply it by $\frac{1}{T_0}$

Example: Find out the Fourier series coefficients of a triangular pulse train by this method.

Soln: From Table 2.3, we note that the truncated triangular pulse has a Fourier transform given as

$$x_{T_0}(t)=\begin{cases}1-\frac{|t|}{T},|t|<T\\ 0,|t|\geq T\end{cases}\Leftrightarrow T\sin c^2(fT)$$

Hence, at a frequency of $f=\frac{n}{T_0}$, this becomes

$$X_{T_0}\left(\frac{n}{T_0}\right)\Leftrightarrow T\sin c^2\left(\frac{n}{T_0}T\right)$$

Multiplying it by , $\frac{1}{T_0}$ we obtain $\frac{1}{T_0}X_{T_0}\left(\frac{n}{T_0}\right)\Leftrightarrow\frac{1}{T_0}.T\sin c^2\left(\frac{n}{T_0}T\right)=\frac{T}{T_0}\sin c^2\left(\frac{n}{T_0}T\right)$

$$\frac{T}{T_0}\sin c^2\left(\frac{n}{T_0}T\right)=\frac{T}{T_0}\left[\frac{\sin\left(\frac{\pi nT}{T_0}\right)}{\left(\frac{\pi nT}{T_0}\right)}\right]^2=\frac{T}{T_0}\frac{\sin^2\left(\frac{\pi nT}{T_0}\right)}{\frac{\pi^2n^2T^2}{T_0^2}}$$

We observe from the above that, for a triangular wave, the Fourier series coefficients decay at the rate of $\left(\frac{n}{T_0}\right)^2$ and they are always positive. This is a faster decay as compared to a similar duration rectangular waveform.

**Objective: To learn power, energy and autocorrelation function of a given signal**.

The energy and power of a signal are representative of the energy or power delivered by the signal when the signal is interpreted as a voltage or current source feeding a 1Ω resistor. The energy content of a signal $x(t)$, denoted by $\mathrm{E}_x$ is defined as

$\mathrm{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$ and the power similarly, can be expressed as

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

A signal is energy-type if $\mathrm{E}_x < \infty$ and is power-type if $0 < P_x < \infty$. A signal can not be, therefore both an energy or a power type signal . For energy type signals, $P_x = 0$ and for power type signals $\mathrm{E}_x = \infty$. Usually all periodic signals (with the exception of $x(t) \equiv 0$) are power-type and have power

$$P_x = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt$$

In the above, $T_0$ is the period of the signal and $\alpha$ is any arbitrary number.

Example: Find out the average power in a periodic sine wave.

Soln: Let the sine wave be represented as

$x(t) = V_m \sin 2\pi ft$

Where $f = \frac{1}{T}$

$$P_x = \frac{1}{T_0} \int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt = \frac{1}{T} \int_0^T (V_m \sin 2\pi ft)^2 dt$$

Thus, $= \frac{V_m^2}{T} \int_0^T \sin^2 2\pi ftdt = \frac{V_m^2}{2T} \int_0^T (1 - \cos 4\pi ft)dt = \frac{V_m^2}{2T}\left[\int_0^T dt - \int_0^T \cos\cos 4\pi ftdt\right] = \frac{V_m^2}{2T}.T$

$= \frac{V_m^2}{2}$

This is because $\int_0^T \cos 4\pi ftdt = 0$

Energy-type Signals

The energy of a signal may be expressed as

$$\mathrm{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \text{ or}$$

$$\mathrm{E}_x = \int_{-\infty}^{\infty} |x(f)|^2 \, df$$

This follows from the fact that the energy of a given signal can not be different whether it is computed in the time domain or in the frequency domain. The equality of the two above expressions is known as Rayleigh's theorem.

Example: Find out the energy contained in a signal given as $x(t) = 10 \sin c 10t$

Soln: It is easier to evaluate the energy in the frequency domain. The spectrum of this signal is $x(t) = 10 \sin c 10t \Leftrightarrow X(f) = \frac{10}{10} \Pi\left(\frac{f}{10}\right) = \Pi\left(\frac{t}{10}\right)$

This is a rectangular pulse in the frequency domain with unit amplitude and bandwidth of 10 units.

Therefore, $\mathrm{E}_x = \int_{-\infty}^{\infty} |x(f)|^2 \, df = \int_{-5}^{5} 1^2 \, df = 10$ units

Relation between convolution and autocorrelation of a given function $x(t)$

We may compute the autocorrelation of an energy-type signal as

$$R_x(\tau) = x(\tau) * x^*(-\tau) = \int_{-\infty}^{\infty} x(t) x^{\bullet}(t-\tau) dt = \int_{-\infty}^{\infty} x(t+\tau) x^{\bullet}(t) dt$$

This is a function of the lag $\tau$ and also gives us the relationship between the autocorrelation and convolution of a given signal. As the signal is correlated with itself for different values of this lag parameter, it is known as autocorrelation. We are trying to find out the degree of similarity between the original waveform and a delayed or advanced version of it.
By setting $\tau = 0$ in the above, we obtain

$$R_x(0) = x(0) * x^*(0) = \int_{-\infty}^{\infty} x(t) x^{\bullet}(t) dt = \int_{-\infty}^{\infty} x(t) x^{\bullet}(t) dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \mathrm{E}_x$$

Let us find out the time-average autocorrelation function and power spectral density of the power type signals. Let us assume that $x(t)$ is a periodic signal with period $T_0$ that has the Fourier series coefficients $\{x_n\}$. The time-average autocorrelation function for such a signal is defined as

$$R_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x^*(t-\tau) dt$$

$$= \lim_{k \to \infty} \frac{1}{kT_0} \int_{-kT_0/2}^{kT_0/2} x(t) x^*(t-\tau) dt = \lim_{k \to \infty} \frac{k}{kT_0} \int_{-T_0/2}^{T_0/2} x(t) x^*(t-\tau) dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) x^*(t-\tau) dt$$

These steps were followed to eliminate the limiting term and to express the autocorrelation function in terms of one period of the signal. The substitution of the Fourier series expansion in the above yields

$$R_x(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t-\tau)dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\sum_{n=-\infty}^{\infty} x_n \exp\left(\frac{j2\pi nt}{T_0}\right)\sum_{m-\infty}^{\infty} x_m^* \exp\left(-\frac{j2\pi n(t-\tau)t}{T_0}\right)dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n x_m^* \exp\left(\frac{j2\pi nt}{T_0}\right)\exp\left(-\frac{j2\pi mt}{T_0}\right)\exp\left(\frac{j2\pi n\tau}{T_0}\right)dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n x_m^* \exp\left(\frac{j2\pi nt}{T_0}\right)\exp\left(-\frac{j2\pi mt}{T_0}\right)\exp\left(\frac{j2\pi n\tau}{T_0}\right)dt$$

$$= \sum_{n=-\infty}^{\infty} x_n x_n^* \exp\left(\frac{j2\pi n\tau}{T_0}\right) = \sum_{n=-\infty}^{\infty} |x_n|^2 \exp\left(\frac{j2\pi n\tau}{T_0}\right)$$

We note that, the autocorrelation function of a periodic signal consists of discrete valued power components located at integral multiples of the fundamental. The power components are proportional to $|x_n|^2$. Taking the Fourier transform of both the sides, we obtain

$$S_x(f) = \Im[R_x(\tau)] = \Im\sum_{n=-\infty}^{\infty} |x_n|^2 \exp\left(\frac{j2\pi n\tau}{T_0}\right) = \sum_{n=-\infty}^{\infty} |x_n|^2 \Im\left[\exp\left(\frac{j2\pi n\tau}{T_0}\right)\right]$$

$$\sum_{n=-\infty}^{\infty} |x_n|^2 \Im\left[\exp\left(\frac{j2\pi n\tau}{T_0}\right)\right] = \sum_{n=-\infty}^{\infty} |x_n|^2 \delta\left(f - \frac{n}{T_0}\right)$$

This $S_x(f)$ gives us the power spectral density of the periodic signal. Power spectral density means the distribution of power of the signal as a function of frequency.

The total power content of the periodic signal is obtained by integrating $S_x(f)$ with respect to frequency . When this is done, the power becomes

$$P_x = \sum_{n=-\infty}^{\infty} |x_n|^2$$

This relation is known as Rayleigh's relation.

## MODULE-II

## AMPLITUDE MODULATION

Modulation of a baseband signal may be viewed as a low pass to band pass conversion. This is usually accomplished by multiplication of the baseband signal with a periodic sinusoidal waveform of a frequency higher known as the carrier than that of the baseband signal. The baseband signal henceforth will be called the modulating signal. Multiplication of the modulating signal with a sinusoidal carrier in the time domain results in a shifting of the spectrum of the modulating signal in the frequency domain. Let the modulating signal be denoted as $m(t)$ and the sinusoidal carrier be $A_c \sin \omega_c t$. Multiplication of the two in the time domain generates a signal $v_{AM}(t)$ expressed as

$$v_{AM}(t) = m(t) A_c \sin \omega_c t$$

If the spectrum of $m(t)$ be $M(f)$, then the product signal $v_{AM}(t)$ has a spectrum given as

$$V_{AM}(f) = \frac{A_c}{2j}\left[M\left(f + f_{c_c}\right) - M\left(f - f_c\right)\right]$$

where $j$ is the complex number equal to $\sqrt{-1}$. The above expression is because of the fact that the spectrum of a pure sinusoid $\sin \omega_c t$ of frequency $f_c$ consists of two impulses centered at $\pm f_c$ with amplitude $\frac{1}{2j}$. In a similar fashion, we note that multiplication of $m(t)$ with a carrier of the form $A_c \cos \omega_c t$ gives us

$$v_{AM}(t) = m(t) A_c \cos \omega_c t$$

The spectrum of this signal takes the form of

$$V_{AM}(f) = \frac{A_c}{2}\left[M\left(f + f_{c_c}\right) + M\left(f - f_c\right)\right]$$

We observe that, the process of multiplication of $m(t)$ with either $A_c \sin \omega_c t$ or $A_c \cos \omega_c t$ has given rise to two new frequency components in the spectrum of the output signal. These two frequencies $f + f_c$ and $f_c - f$ are called the upper side band (USB) and the lower side band (LSB) respectively. The process of generation of these two side bands along with the carrier is known as double side band with carrier (DSB plus C). The expression for DSB with full carrier is

$$v_{DSB+C}(t) = m(t) + m(t) A_c \sin \omega_c t = m(t)\left[1 + A_c \sin \omega_c t\right]$$

As we may further observe, it can be generated by a multiplier and adder circuit. This is illustrated in Fig. L9.1.

For example: $L = 3$, and $m = 7$ The terms $(L-1) = 2$. The pulse $q(t-kT)$ is present for the instants from $(-2)$ to $0$. This lasts for, hence 3 symbol intervals. However, the shifted pulse $q(t+\tau-kT)$ lasts from $k=1$ to $k = m - L = 7 - 3 = 4$th instant which has saturated to ½ as the pulse at the 7th signalling interval may originate at this, may have the 6th pulse as its only or the 5th pulse may be the pulse two intervals earlier. Hence, all the values of $q(t+\tau-kT)$ would have saturated to ½ from -2 to 4th signalling interval whereas the original pulse

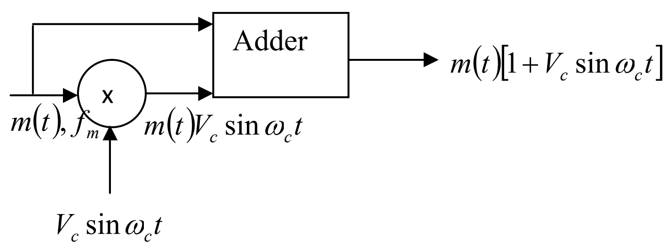



Fig. L 9.1 Conceptual generation of DSB with full carrier type of AM signal

For a sinusoidal modulating signal, the instantaneous amplitude of the carrier becomes

$V_c + V_m \sin\omega_m t$ as the modulating signal sits atop the amplitude of the carrier. As we are interested in the instantaneous amplitude of the carrier as it should change in accordance with the amplitude of the modulating signal, the overall modulated signal looks like

$$v_{DSB+C}(t) = V_m \sin\omega_m t + V_c \sin\omega_c t . V_m \sin\omega_m t = V_c\left[1 + \frac{V_m}{V_c}\sin\omega_m t\right]\sin\omega_c t$$

We define the modulation index or the depth of modulation of this type of AM signal is defined as

$$m_a = \frac{V_m}{V_c}$$

The ratio of the peak amplitudes of the carrier and the modulating signal and it has a maximum value of unity. Usually, the value of $m_a \leq 1$, in order for an envelope detector to work at the receiver. If $m_a = 1$, we understand it as 100% modulated signal and for a value of $m_a > 1$, we realize an overmodulated signal. For standard AM broadcast, the value of modulation index is 30%. Depending on the amplitude level of the modulating signal, a modulator may be a low level modulator or a high level modulator. A low level modulator may be constructed by injecting the modulating signal either to the base or the emitter of a transistor. Let us study such a modulator.

is zero.

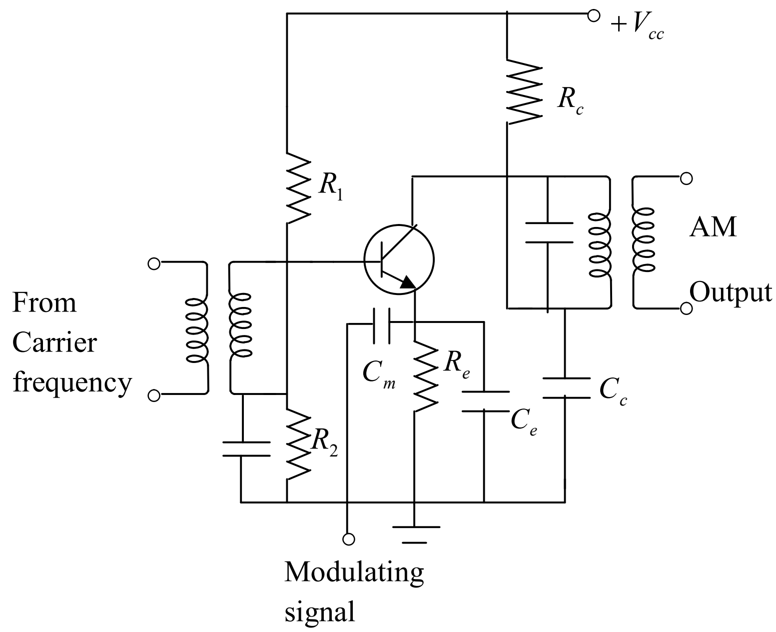



Fig.L 9.2 A BJT amplifier with emitter modulation circuit to generate DSB plus C

In Fig.L. 9.2, the dc bias condition is set up by the voltage divider $R_1$ and $R_2$, the emitter resistor $R_e$, collector resistor $R_c$ and the supply voltage $V_{cc}$. The ac voltage gain of the BJT amplifier depends on its quiescent emitter current. As the modulating signal has been injected into the emitter, the instantaneous emitter current becomes

$$i_E = I_E + K_1 V_m \cos \omega_m t$$

where $I_E$ is the quiescent value of the emitter current and $K_1$ is a constant. Amplitude modulation results if $K_1 V_m$ is smaller than $I_E$. As the voltage amplification is a function of the total emitter current, we get

$$A_v = K_2 i_E = K_2 \left(I_E + K_1 V_m \cos \omega_m t\right)$$

where $K_2$ is another constant. The input to the amplifier is the carrier voltage coupled through a transformer, the output voltage of this circuit is

$$V_0 = A_v V_c \cos \omega_c t = K_2 \left(I_E + K_1 V_m \cos \omega_m t\right) \cos \omega_c t$$

We can observe that, amplitude modulation has been achieved. The tuned circuit present at the collector allows the two side bands to pass through and suppresses other harmonics from appearing at the output. This constitutes a band pass filter with center frequency around the carrier frequency with a pass band of $2f_m$.

A low level modulation is also achieved by injecting the modulating signal to the base of the transistor. The circuit for achieving this is illustrated in Fig. L 9.3.

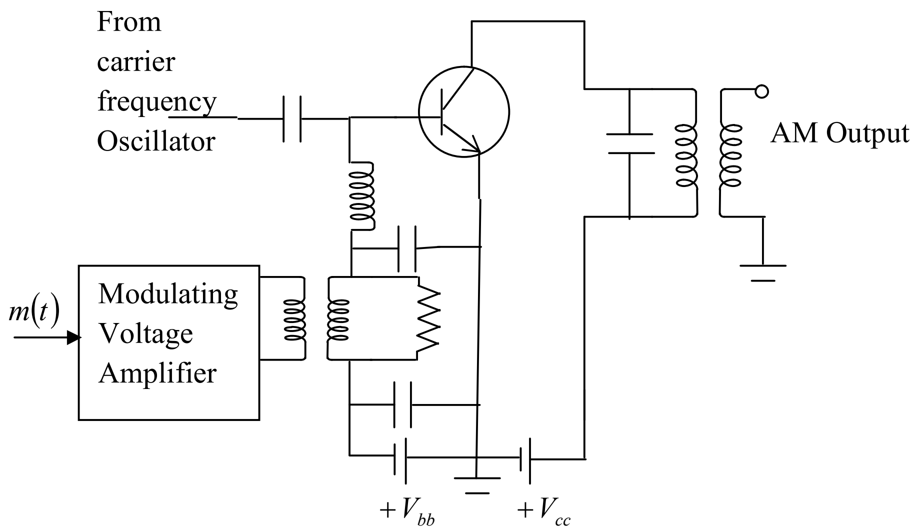



Fig. L 9. 3 A BJT amplifier with base modulation circuit to generate DSB plus C

Another circuit to accomplish DSB plus C generation is the switching modulator illustrated in Fig.L 9. 4.

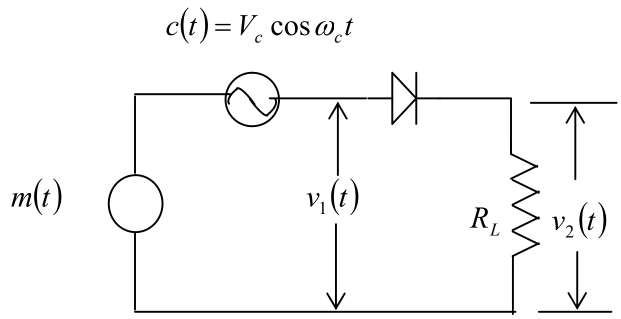



Fig. L 9.4 A Switching Modulator

In this circuit, we assume that the carrier applied to the diode is larger than the modulating signal in amplitude. It is further assumed that the diode is an ideal switch which implies that for the forward bias condition corresponding to $c(t) > 0$, it shows zero resistance. The transfer characteristic of the diode-load resistor may be modeled as piece wise linear. This means

$$v_2(t) = \begin{cases} v_1(t), & c(t) > 0 \\ 0, & c(t) < 0 \end{cases}$$

where $v_1(t) = m(t) + V_c \cos \omega_c t$. We observe from the above that, the output voltage $v_2(t)$ varies periodically between the voltage $v_1(t)$ and zero with a frequency of $f_c$. The output voltage, may alternatively be expressed as

$$v_2(t) \approx [m(t) + V_c \cos \omega_c t] g(t)$$

where $g(t)$ is viewed as a periodic pulse train with unity amplitude and a duty cycle of 50%, the time period being equal to $T_0 = \frac{1}{f_c}$. The Fourier series expansion of this pulse train gives us

$$g(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos[\omega_c t (2n-1)]$$

Substitution of this in the above expression gives rise to two components. The first term is $\frac{V_c}{2}\left(1 + \frac{4}{\pi V_c} m(t)\right) \cos \omega_c t$ is the desired DSB plus C component. The second term that contains all harmonics are filtered out by the use of a band pass filter with a center frequency of $f_c$ with a bandwidth of $2f_m$.

Square Law Modulator

A square law modulator is shown in Fig. L 9.5. This uses the nonlinear property of an active device like a diode, BJT etc. The modulating signal is relatively weak. The output of the device can be related to the input as

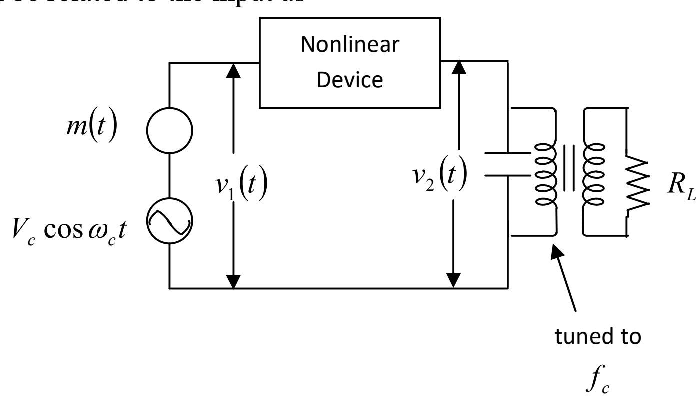



Fig.L 9.5 A square law modulator that employs a nonlinear device

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t)$$

where $a_1$ and $a_2$ are constants. The input voltage is expressed as

$$v_1(t) = m(t) + V_c \cos \omega_c t$$

Hence, the output voltage becomes

$$v_2(t) = a_1[m(t) + V_c \cos\omega_c t] + a_2[m(t) + V_c \cos\omega_c t]^2$$

Expansion of the second term in the above gives us

$$v_2(t) = a_1[m(t) + V_c \cos\omega_c t] + a_2[m(t) + V_c \cos\omega_c t]^2$$

$$a_2\left[m^2(t) + 2V_c m(t)\cos\omega_c t + V_c^2 \cos^2\omega_c t\right] = a_2\left[m^2(t) + 2V_c m(t)\cos\omega_c t + \frac{1}{2}(1 + \cos 2\omega_c t)\right]$$

**LECTURE-10**

### L 10.1 High Level Modulator

All the transmitters employing the previous circuits are known as low level modulators. This is because the amplitude of the modulating signal is rather small that may come from a microp phone or a typical video camera like the vidicon. Amplification of the modulated signal takes place after these circuits. Hence such circuits are known as low level transmitters. For the high level modulation, the modulating signal is amplified first before it amplitude modulates the carrier. This is usually carried out in class-C power amplifiers. This is because, as the modulating signal has been already amplified, it can not drive linear power amplifiers. Such a high level modulator employing a class-C power amplifier is shown in Fig. L 10.1.

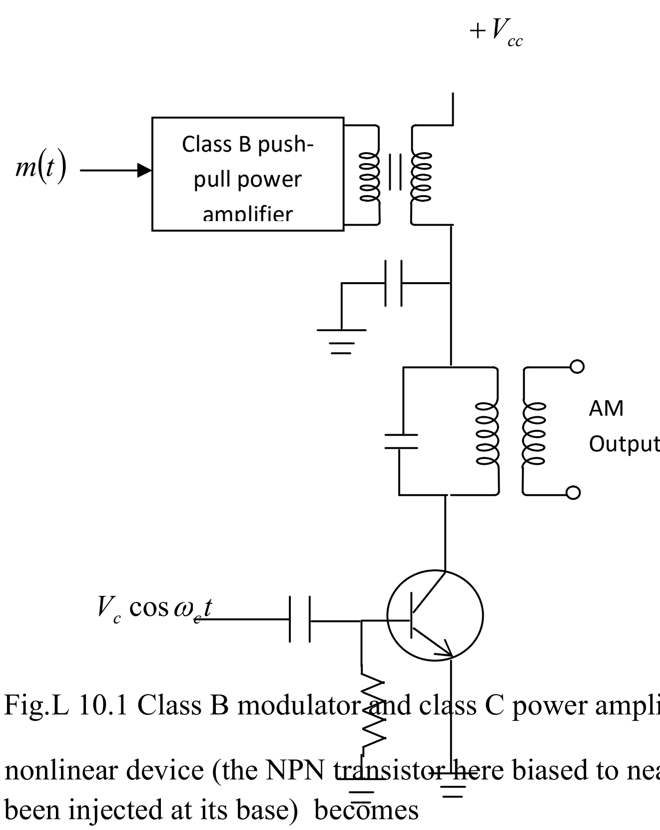



Fig.L 10.1 Class B modulator and class C power amplifier

The output of the nonlinear device (the NPN transistor here biased to near cut off by the carrier which has been injected at its base) becomes

$$v_2(t) = a_1[m(t) + V_c \cos\omega_c t] + a_2\left[m^2(t) + 2V_c m(t)\cos\omega_c t + \frac{1}{2}(1 + \cos 2\omega_c t)\right] \qquad \text{(L 10.1)}$$

considering only upto the second term in the power series expression for the output of a nonlinear power amplifier

We observe from the above that,

$$a_1 V_c \cos\omega_c t + 2a_2 m(t) V_c \cos\omega_c t = V_c(a_1 + 2a_2 m(t))\cos\omega_c t \qquad \text{(L 10.2)}$$

Is the desired DSB with carrier. To realize demodulation with simple, low cost demodulators such as an envelope detector, we have to ensure that

$$\left|\frac{2a_2}{a_1}\right| < 1$$

The component with $\cos 2\omega_c t$ has been rejected by the tuned circuit (band pass filter with a centre frequency equal to the carrier frequency) connected to the collector of the power amplifier and hence does not appear at the output of the modulator.

## L 10.2 POWER IN AN AM SIGNAL

A conventional (DSB with full carrier) AM signal expressed as

$$v_{AM}(t) = V_c[1 + m_a \cos\omega_m t]\cos\omega_c t \qquad \text{(L 10.3)}$$

corresponding to a modulating signal expressed as $V_m \cos\omega_m t$. This is otherwise known as tone modulation.

From (L 10.3), it is observed that,

$$v_{AM}(t) = V_c \cos\omega_c t + \frac{m_a V_c}{2}[\cos(\omega_c + \omega_m)t + \cos(\omega_c - \omega_m)t] \qquad \text{(L 10.4)}$$

The first term in (L 10.4) gives us a power of $\frac{V_c^2}{2}$. This is because a periodic sine wave with unit amplitude such as a carrier has an time-averaged power equal to ½ W.

Both the second and the third terms give us equal powers of $\frac{m_a^2 V_c^2}{8}$ as these are also sinusoidal waveforms. The total power, hence becomes in a DSB plus carrier type of AM waveform,

$$P_t = \frac{V_c^2}{2} + \frac{m_a^2 V_c^2}{8} + \frac{m_a^2 V_c^2}{8} = \frac{V_c^2}{2} + \frac{m_a^2 V_c^2}{4} = \frac{V_c^2}{2}\left(1 + \frac{m_a^2}{2}\right) \qquad \text{(L 10.5)}$$

We observe that from (L 10.5), out of this total power, the carrier power is $P_c = \frac{V_c^2}{2}$ whereas the modulating signal gives us a power of $\left(\frac{m_a V_c}{2}\right)^2$.

This carrier power represents a wastage of power as it does not convey any useful information. If the modulation index has a value of 1, then the total transmitted power is 1.5 $\frac{V_c^2}{2}$. If we choose not to transmit the carrier power, then we actually transmit a power of 0.5 which accounts for a power saving of 66%. This is so as the carrier does not contain any useful information about the modulating signal. If the modulating signal is any arbitrary signal $m(t)$, then its average power becomes $\overline{m^2(t)}$ and the total power in a DSB plus carrier type of AM waveform becomes $P_c\left(1+\overline{m^2(t)}\right)$. Similarly, the power in a DSBSC type of AM waveform is $P_c\overline{m^2(t)}$. The SSBSC type of AM waveform will have a power content of $1/2P_c\overline{m^2(t)}$.

# LECTURE-11

Objective:

To learn DSB-SC modulation/demodulation techniques

- (a) Balanced Modulator Circuit

In a DSB-SC form of amplitude modulation, carrier is suppressed as it does not convey any information. This carrier suppression is accomplished in a number of ways. We start with a balanced modulator circuit. This is realized by BJT/FETs or devices possessing nonlinear characteristics. Such a circuit is shown in Fig. L 11.1.

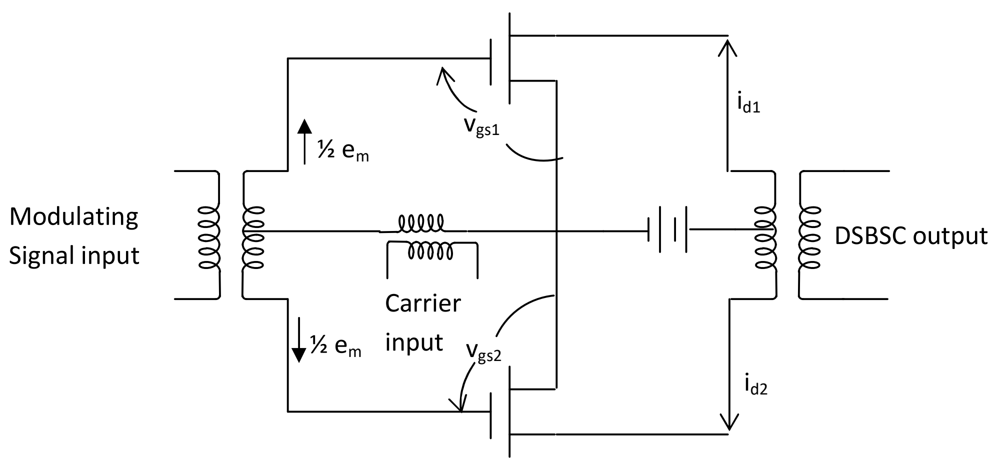



Fig. L 11.1 A balanced modulator circuit realized with two FETs

Any circuit that produces the product of two input waveforms (the modulating signal and the carrier) is a balanced modulator. The FET is used here as it has a transfer characteristic which is nonlinear, so that the output contains a term equal to the product of the input voltages, besides other cross terms. The transfer characteristic of the FET is almost parabolic and may be approximated as

$$i_d = I_0 + av_{gs} + bv_{gs}^2 \quad (11.1)$$

where $I_0$ is the current for zero gate-source voltage, and $a, b$ are constants. Since the drain currents $i_{d1}$ and $i_{d2}$ flow in the opposite directions in the primary winding of the output transformer, the effective primary current $i_p$ is

$$\begin{aligned} i_p &= i_{d1} - i_{d2} = a\left(v_{gs1} - v_{gs2}\right) + b\left(v_{gs1}^2 - v_{gs2}^2\right) \\ &= a\left(v_{gs1} - v_{gs2}\right) + b\left(v_{gs1} - v_{gs2}\right)\left(v_{gs1} + v_{gs2}\right) \end{aligned} \qquad (11.2)$$

This becomes equal to, upon application of Kirchoff's law to the input loops of Fig. L 11.1,

$$v_{gs1} = \frac{1}{2}e_m + e_c \text{ and } v_{gs2} = \frac{1}{2}e_m - e_c \qquad (11.3)$$

We obtain,

$$i_p = a\left(e_m\right) + b\left(2e_c\right)\left(e_m\right) \qquad (11.4)$$

The RF output transformer rejects the low-frequency term like $e_m$ passing only the product $2be_ce_m$ which is the desired DSBSC signal. However, generation of DSBSC by this circuit requires the two FETS matched completely with respect to $I_0, a$ and $b$. Otherwise, residual components would appear at the output which obviously is not the desired modulated waveform. These days the BMs are available in the integrated circuit (IC) form. Motorola's AN531 is one such IC.

- (b) Chopper Amplifier based modulator

Fig. L 11.2 (a)

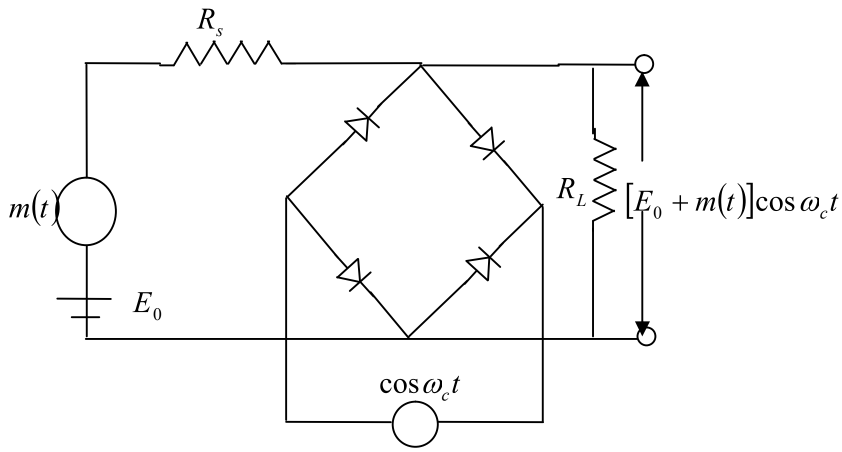



Fig. L 11. 2 Chopper balanced modulator

In Fig. L 11.2, chopping of the signal is accomplished by the diode bridge at a rate equal to the carrier frequency. The signal applied to the bridge is the message signal plus the dc bias. All four diodes of the bridge conduct during the positive half cycle of the carrier thereby giving no output voltage and none of them conduct during the negative half cycles of the carrier alternately which makes the signal becoming available across the load resistance. The carrier is prevented at the output by means of a tuned circuit.

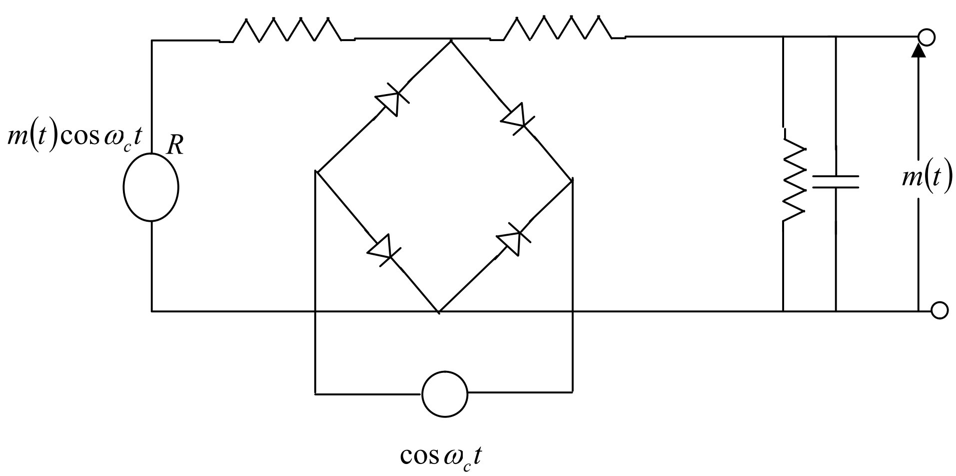



Fig. L 11.3 Demodulation of the DSBSC signal produced by Fig. L 11. 2(a)

For demodulation of the DSBSC signal, we need to multiply $m(t)\cos\omega_c t$ by a synchronously generated carrier $\cos\omega_c t$. The same circuits as those used for modulation can be used for demodulation. However, the demodulating circuit differs from the modulator in that the output of the demodulator should contain a low pass filter whereas the modulator has a bandpass filter at its output. The low pass filtering is provided by the RC combination as shown in the above figure. The demodulation may be accomplished by multiplying the modulated signal by any periodic signal of frequency $\omega_c$.If $\rho(t)$ is any periodic signal of frequency $\omega_c$, then it has a Fourier series $\Phi(f)$ given as

$$\rho(t) \leftrightarrow \sum_{n=-\infty}^{\infty} \Phi_n \delta(f - nf_c)$$

It is apparent that, if the modulated signal $m(t)\cos\omega_c t$ is multiplied by this periodic signal $\rho(t)$, the corresponding spectrum becomes

$$m(t)\cos\omega_c t \rho(t) \leftrightarrow \frac{1}{2}[M(f - f_c) + M(f + f_c)] * \sum_{n=-\infty}^{\infty} \Phi_n \delta(f - nf_c)$$

$$\leftrightarrow \frac{1}{2}\sum_{n=-\infty}^{\infty} \Phi_n M\{[f - (n+1)f_c] + M[f - (n-1)f_c]\}$$

From the above, it is observed that the resultant spectrum contains a term $M(f)$ which can be filtered out by a low pass filter.

Another form of the balanced modulator is shown in Fig. L 11.2 (b).

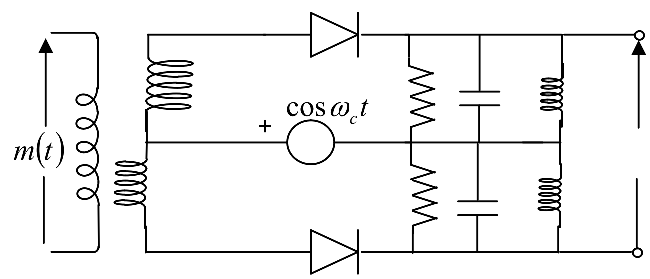


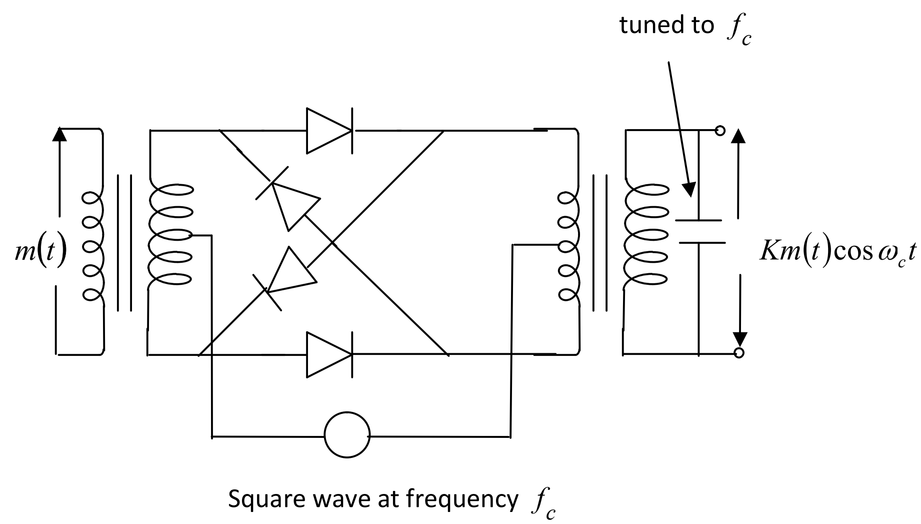

Fig. L 11. 4 Another realization of a balanced modulator

Modulation is achieved by using nonlinear devices. A semiconductor diode is a nonlinear device. A nonlinear device such as a diode may be approximated by a power series like
$i = av + bv^2$

Transistors and vacuum tubes also exhibit similar relationships between the input and the output under large signal conditions. To analyze this circuit, we consider the nonlinear circuit element in series with the resistance $R$ as a composite nonlinear element whose terminal voltage $v$ and the current $i$ are related as above. The voltages $v_1$ and $v_2$ are given as
$v_1 = \cos\omega_c t + m(t)$ and $v_2 = \cos\omega_c t - m(t)$
The currents $i_1$ and $i_2$ are given as
$i_1 = av_1 + bv_1^2 = a[\cos\omega_c t + m(t)] + b[\cos\omega_c t + m(t)]^2$ and
$i_2 = a[\cos\omega_c t - m(t)] + b[\cos\omega_c t - m(t)]^2$
Hence the output voltage is given by

$$v_0 = i_1 R - i_2 R = 2R[2bm(t)\cos\omega_c t + am(t)]$$

The signal $am(t)$ in this equation can be filtered put by using a bandpass filter tuned to $\omega_c$ at the output terminals. Semiconductor diodes are conveniently used for the nonlinear circuit elements in this circuit. All of the modulators discussed above generate a suppressed-carrier amplitude modulated signal and are known as balanced modulators.




Fig, L 11.5. Ring Modulator that uses a centre tapped transformer at input as well as the output

The diodes in Fig.L 11.5 form a ring as they all point in the same way. They are controlled by a square wave $c(t)$ of frequency equal to carrier frequency $f_c$ which is applied in a longitudinal manner by means of two centre-tapped transformers. Under the assumptions of a perfect centre tap and identical diodes, there would be no leakage of modulation frequency into the modulator output. Let us assume the diodes to be ideal. On the positive half cycle of the square wave serving as the carrier, the top and bottom diodes become 'on' and the signal $m(t)$ passes on to the output. Similarly, during the negative half cycles of the carrier, the diagonal diodes become 'on' switching off the top and bottom diodes. Hence the message signal passes on to the output, however with a negative polarity. Let us find out the kind of modulated waveform at the secondary output of the output transformer.

The square wave has a Fourier series given as

$$c(t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\left[2\pi f_c t(2n-1)\right]$$

The ring modulator output is, therefore

$$s(t)=m(t)c(t)$$

$$=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\cos\left[2\pi f_c t(2n-1)\right]m(t)$$

There is no output from the modulator at the carrier frequency, that is the modulator output consists entirely of modulation products. The ring modulator sometimes is referred to as the double-balanced modulator because it is balanced with respect to the message signal as well as the carrier.

Under the assumption of the message signal being bandlimited to $-f_m \leq f \leq f_m$, the spectrum of the modulator output consists of sidebands around each of the odd harmonics of the square wave carrier as shown in Fig. Here it has been assumed that $f_c > f_m$ so that sideband overlapping is avoided which arises when sidebands belonging to adjacent harmonic frequencies $f_c$ and $3f_c$ overlap with each other. A bandpass filter with a centre frequency $f_c$ and bandwidth $2f_m$ at the output would select the sidebands centered around $f_c$ and reject all other components.

## LECTURE-12 DEMODULATION OF AM SIGNALS

### (a) Demodulation of DSB with full carrier type of modulated signals

Demodulation is the process of recovery of the original message signal embedded in the AM wave. This is accomplished by the demodulator circuit in the receiver. The simplest demodulator is a rectifier followed by a low pass filter which is called diode detector.

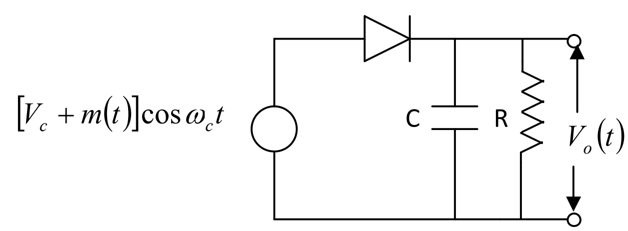



Fig. L 12.1 Envelope detector for conventional AM systems

This circuit is called so as it responds to the envelope of the incoming AM signal. On the positive half cycle, the diode conducts and the capacitor C charges to the peak value of the rectified voltage. As the incoming signal falls below this value the diode becomes non conducting. This is due to the fact that the anode side voltage of the diode is less than the cathode side voltage. Thus, the capacitor tends to hold the previously acquired peak value. The capacitor discharges through the resistor at a slow rate. During the next positive half cycle, the input signal becomes greater than the capacitor voltage and the diode starts conducting again allowing the capacitor to charge up to the immediate peak value. The capacitor discharges slowly during the off period of the diode which results in a small change in its output voltage. During each positive half cycle, the capacitor charges to the peak value of the incoming signal and holds this voltage until the next positive cycle. The time constant RC of the output circuit is adjusted in such a manner that the exponential decay of the capacitor voltage during the discharge period will follow the envelope approximately. The output voltage now has a ripple component at $\omega_c$ which is filtered out by another low pass filter.

The instantaneous AM signal is

$$v_{AM}(t) = V_c(1 + m_a \cos\omega_m t)$$

At any time instant, $t = t_0$, the slope of the envelope is given by

$$\left.\frac{dv_{AM}(t)}{dt}\right|_{t=t_0} = -m_a\omega_m V_c \sin\omega_m t_0$$

At that particular time, the envelope is given as

$$v_{AM}(t_0) = V_c(1 + m_a \cos\omega_m t_0)$$

Let $t_0$ be the time instant when the capacitor C starts discharging. At any subsequent time $t$, the decayed capacitor voltage becomes

$$v(t) = V(t_0)\exp^{-\left(\frac{t-t_0}{RC}\right)}$$

At $t = t_0$, the rate of change of decay is

$$\left.\frac{dv(t)}{d(t-t_0)}\right|_{t=t_0} = -\frac{V(t_0)}{RC} = -\frac{V_c(1 + m_a \cos\omega_m t_0)}{RC}$$

If clipping of the negative peaks of the modulating signal is to be avoided, then at $t = t_0$, the slope of the decayed capacitor voltage must be equal to or less than that of the modulated carrier. This is equivalent to saying that,

$$-\frac{V_c(1 + m_a \cos\omega_m t_0)}{RC} \le -m_a\omega_m V_c \sin\omega_m t_0$$

or, $\dfrac{1}{RC} \ge \dfrac{m_a\omega_m \sin\omega_m t_0}{1 + m_a \cos\omega_m t_0}$

This gives us an upper limit for the circuit time constant as

$$RC \le \frac{1}{\omega_m}.\left(\frac{1}{\dfrac{m_a\omega_m \sin\omega_m t_0}{1 + m_a \cos\omega_m t_0}}\right)$$

Making an maximization of RHS, the term $\dfrac{m_a\omega_m \sin\omega_m t_0}{1 + m_a \cos\omega_m t_0}$ is maximum when

$\cos\omega_m t_0 = -m_a$ which implies that $\sin\omega_m t_0 = \sqrt{1 - m_a^2}$

Substitution of the values of the above yields

$$RC \le \frac{1}{\omega_m}.\frac{\sqrt{1 - m_a^2}}{m_a}$$

The above equation indicates that, for 100% modulation, the product RC should be zero which is not practical. In practice, it is found that for

$$RC \le \frac{1}{\omega_m m_a}$$

the distortion in the diode demodulator output is not excessive. The highest frequency that can be detected by this circuit is

$$\omega_{mHigh} = \frac{1}{RCm_a}$$

**(b) Demodulation of suppressed carrier type of modulated signals**

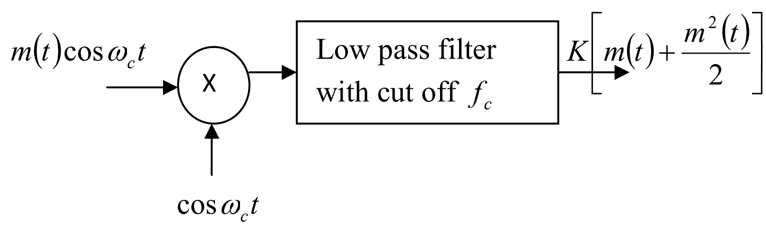



Fig. L 12.2 A synchronous/ coherent demodulator for DSBSC signals

**(i) Costas Loop**

This as shown in Fig. L 12.3 consists of two coherent detectors. A voltage controlled oscillator initially adjusted to operate at the correct suppressed carrier frequency, $f_c$, assumed to be known a priori, supplies the locally generated carrier to the two coherent detectors-to one of them directly and to the other through a -90$^0$ phase shifter. The top coherent detector receives the $\cos(\omega_c t+\theta)$ directly from the voltage controlled oscillator (VCO). The bottom balanced modulator has a carrier of the form $\sin(\omega_c t+\theta)$ obtained by feeding the VCO output through a 90$^0$ phase shifter. The incoming DSBSC signal $E_c m(t)\cos(\omega_c t)$is fed as the other input to both of the balanced modulators. Suppose the carrier phase error is zero which means the phase offset $\theta$ between the incoming carrier and the locally generated carrier is zero. Then the output of the I-channel is $\frac{1}{2}E_c m(t)$ and that of the Q-channel is zero. The I-channel output is taken as the demodulated signal. Now under a practical situation, there exists a finite phase offset between the two carriers. Then, for such a case the I-channel produces an output proportional to $\frac{E_c}{2}m(t)\cos\theta$ while that of the Q-channel is $\frac{E_c}{2}m(t)\sin\theta$. Both of these outputs have been shown to be fed to the phase discriminator which consists of a multiplier followed by a low pass filter. For values of $\theta$ quite small, we have $\cos\theta \approx 1$ and $\sin\theta \approx 0$. The low pass filter used in the phase discriminator has a cut off frequency of the order of a few Hertz, gives a dc voltage proportional to $\theta$ at its output since variations in $\theta$ will be very slow as compared to the variations in $m^2(t)$. Thus we have a dc voltage that has the same polarity as $\theta$ and is proportional to it. This changes the frequency of oscillation of VCO in such a way so as to lock it to $f_c$, thereby keeping the phase offset within very small values.

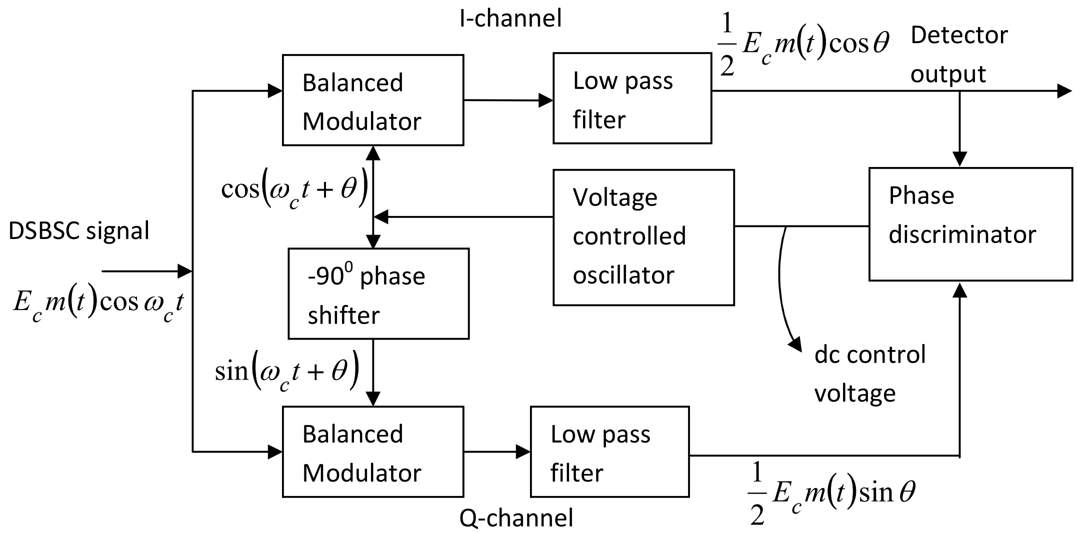



Fig. L 12.3 Costas loop for demodulation of DSBSC signals

The Costas loop provides a good practical solution to achieve phase synchronism common to coherent detection. However, it suffers from one major disadvantage-the $180^0$ phase ambiguity of the demodulated signal. Suppose in stead of receiving $E_c m(t)\cos(\omega_c t)$ we have $-E_c m(t)\cos(\omega_c t)$. The output of the multiplier used in the phase discriminator produces an output proportional to $E_c^2 m^2(t)\theta$, it is insensitive to the polarity of the incoming signal. Under the locked conditions of the phase discriminator, we are not certain about the polarity of the demodulated signal; whether it is $m(t)$ or $-m(t)$. However, for demodulating audio signals, this does not pose a serious problem as our ears are insensitive to polarity of the demodulated signal. For video signals, a demodulated signal with negative polarity reproduces an inverted picture in the receiver which is obviously very objectionable. Similarly, for polar data also this phase ambiguity issue would damage the data as ‘1’ becomes ‘0’ and vice-versa. The phase control of the loop ceases for the condition of no modulation present at the input. However, this is not a serious problem as the loop establishes the lockup condition very fast.

**(ii) Squaring loop**

Another realization of a DSBSC demodulator is shown in Fig. L 12.4. This is known as a squaring loop.

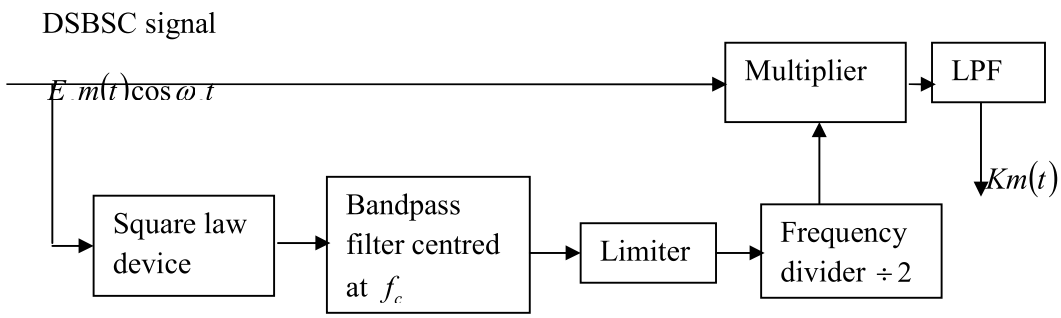



Fig. L 12.4 A squaring loop

## Single Sideband Modulation

Hilbert Transform

The Hilbert transform a time function is obtained by shifting all frequency components by $90^0$. It is, therefore represented by a linear system having a transfer function $H(f)$ as shown in the figure below

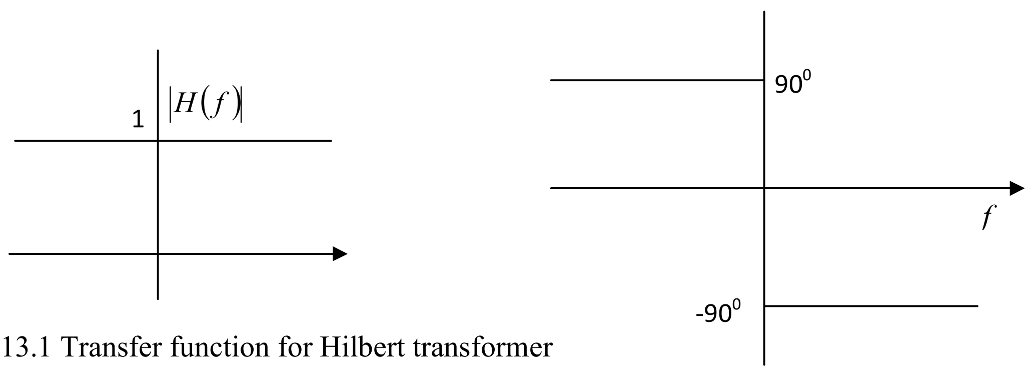



Fig. L 13.1 Transfer function for Hilbert transformer

We note that the phase function is odd. The positive frequency components get a $-90^0$ phase shift whereas the negative frequencies undergo a $90^0$ phase shift. The system function is given as

$H(f) = -j\,\mathrm{sgn}(f)$ corresponding to an impulse response of

$$h(t) = \frac{1}{\pi t}$$

The SSB signal may be generated by passing a DSBSC modulated signal through a band-pass filter of transfer function $H_u(f)$. Let us find out this $H_u(f)$. We know that a DSBSC signal is expressed as

$$s_{DSBSC}(t) = E_c m(t)\cos 2\pi f_c t$$

This is a bandpass signal containing only the in phase component. The low pass complex envelope of the DSBSC modulated signal is given as

$$\tilde{s}_{DSBSC}(t) = E_c m(t)$$

The SSB modulated signal is also a bandpass signal. However, unlike the DBSC modulated signal, it has a quadrature as well as an inphase component. Let the low pass signal $\tilde{s}_u(t)$ denote the complex envelope of $s_u(t)$. Hence,

$$s_u(t) = \text{Re}[\tilde{s}_u(t)\exp(j2\pi f_c t)]$$

We next proceed to find out the low pass complex equivalent $\tilde{s}_u(t)$. To do so, the bandpass filter transfer function is replaced by a an equivalent low pass filter of transfer function $\tilde{H}_u(f)$ as shown in Fig. From the Fig. we observe that

$$\tilde{H}_u(f) = \begin{cases} \frac{1}{2}[1+\text{sgn}(f)], & 0 < f < f_m \\ 0 & \text{elsewhere} \end{cases}$$

The DSBSC modulated signal is replaced by its complex envelope. The spectrum of this is

$$\tilde{S}_{DSBSC}(f) = E_c M(f)$$

The desired complex envelope $\tilde{s}_u(t)$ is determined by evaluating the inverse Fourier transform of the product of $\tilde{H}_u(f)\tilde{S}_{DSBSC}(f)$. Thus,

$$\tilde{H}_u(f)\tilde{S}_{DSBSC}(f) = \frac{E_c}{2}M(f)[1+\text{sgn}(f)]$$

Let us have a signal $\hat{m}(t)$ such that $\hat{m}(t) \Leftrightarrow -j\,\text{sgn}(f)M(f)$

Thus,

$$\tilde{s}_u(t) = \frac{E_c}{2}[m(t) + j\hat{m}(t)]$$

Accordingly, the mathematical expression for the SSB modulated wave is

$$s_u(t) = \frac{E_c}{2}[m(t)\cos 2\pi f_c t - \hat{m}(t)\sin 2\pi f_c t]$$

This equation tells us that, except for a scaling factor, a modulated wave containing only an upper sideband has an inphase component equal to the message signal $m(t)$ and a quadrature component equal to $\hat{m}(t)$, the Hilbert transform of $m(t)$.

From the foregoing we may note that, when the objective is to retain the lower sideband only, the transfer function of the bandpass filter needs to be modified to

$$\tilde{H}_l(f) = \begin{cases} \frac{1}{2}[1-\text{sgn}(f)], & -f_m < f < 0 \\ 0 & \text{elsewhere} \end{cases}$$

Thus, the output of this bandpass filter in response to the complex envelope of the DSBSC modulated signal becomes

$$\tilde{H}_l(f)\tilde{S}_{DSBSC}(f) = \frac{E_c}{2}M(f)[1-\text{sgn}(f)]$$

that gives us

$$\tilde{s}_l(t) = \frac{E_c}{2}\left[m(t) - j\hat{m}(t)\right]$$

Accordingly, the mathematical expression for the SSB modulated wave is that contains the lower sideband only is

$$s_l(t) = \frac{E_c}{2}\left[m(t)\cos 2\pi f_c t + \hat{m}(t)\sin 2\pi f_c t\right]$$

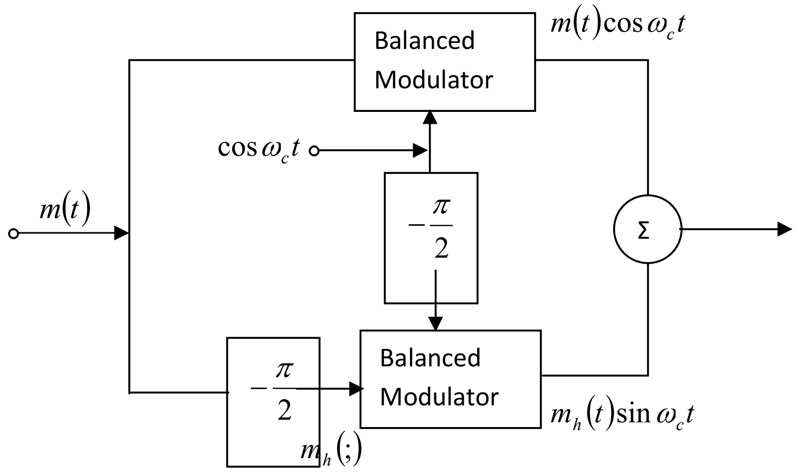



Fig. L 13.2 Phase shift method of generation of SSB-SC signal

The SSB signal as generated by Fig. L 13.2 has a waveform expressed as

$$\varphi_{SSB}(t) = m(t)\sin\omega_c t + m_h(t)\cos\omega_c t$$

where $m_h(t)$ is the signal obtained by shifting the phase of each frequency component of $m(t)$ by $\frac{\pi}{2}$.

(b) Weaver's method of SSB-SC generation

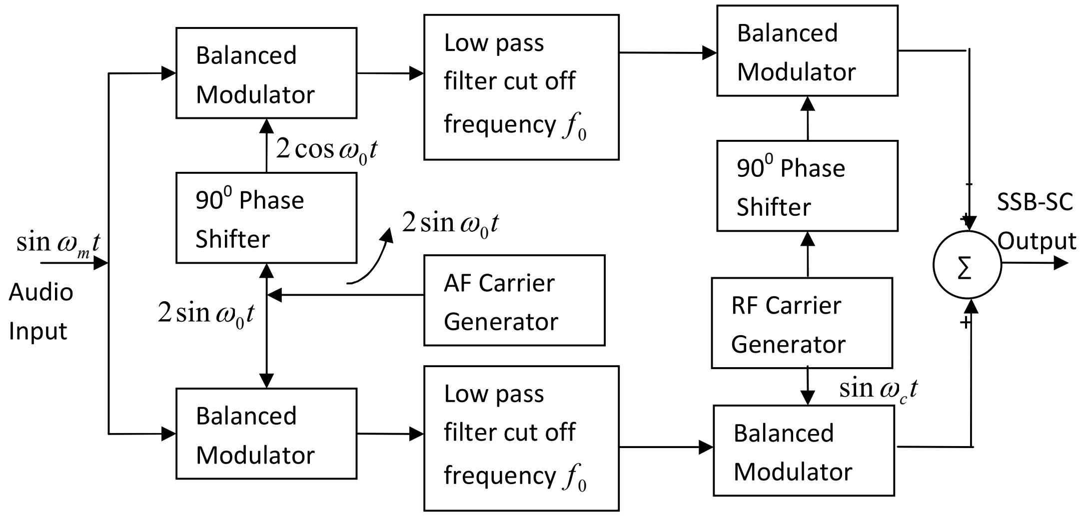



Fig. L 13.3 SSBSC generation by Weaver's method

The Weaver's method modifies the phasing method to rid of the design issues arising in wideband phase shifters. It uses an audio frequency sub carrier at a frequency of $f_0$. Let us find out the expression for the summer output as shown in Fig.

The top left balanced modulator produces an output which is given as

$$2\sin\omega_m t\cos\omega_0 t = [\sin(\omega_0 - \omega_m)t - \sin(\omega_0 + \omega_m)t]$$

The low pass filter with a cut off frequency of $f_0$ rejects the higher frequency term given by $(\omega_0 + \omega_m)$. Therefore, the output of the top right balanced modulator is expressed as

$$\sin[(\omega_0 - \omega_m)t]\cos(\omega_c t) = \frac{1}{2}[\sin(\omega_c - \omega_0 + \omega_m)t - \sin(\omega_c + \omega_0 - \omega_m)t]$$

The bottom left balanced modulator produces an output which is given as

$$2\sin\omega_m t\sin\omega_0 t = [\cos(\omega_0 - \omega_m)t + \cos(\omega_0 + \omega_m)t]$$

The low pass filter with a cut off frequency of $f_0$ rejects the higher frequency term given by $(\omega_0 + \omega_m)$. Therefore, the output of the bottom right balanced modulator is expressed as

$$\cos[(\omega_0 - \omega_m)t]\sin(\omega_c t) = \frac{1}{2}[\sin(\omega_c - \omega_0 + \omega_m)t - \sin(\omega_c + \omega_0 - \omega_m)t]$$

Hence, the output of the summing amplifier becomes

$$\frac{1}{2}[\sin(\omega_c - \omega_0 + \omega_m)t - \sin(\omega_c + \omega_0 - \omega_m)t] + \frac{1}{2}[\sin(\omega_c - \omega_0 + \omega_m)t - \sin(\omega_c + \omega_0 - \omega_m)t]$$

$$= \sin(\omega_c - \omega_0 + \omega_m)t - \sin(\omega_c + \omega_0 - \omega_m)t$$

Hence, the modulator generates the USB-SC corresponding to a carrier frequency of $(f_c - f_0)$ or the LSB-SC corresponding to a carrier frequency of $(f_c + f_0)$.

The Weaver's method has certain advantages such as:

- No need for any sideband suppression filter
- No need for any wideband phase shifter

- As the phase shifters are designed for a single frequency, they are extremely simple and cheap.
- No need for frequent adjustments
- Easy to change from USB-SC to LSB-SC and vice versa at the summing junction output.

**Demodulation of SSB signals**

Can be accomplished by any synchronous/ coherent kind of demodulators discussed earlier.

## Vestigial Sideband Modulation

This is a compromise between the bandwidth conserving feature of a typical SSBSC modulation and demodulation simplicity of the conventional AM signals. This is widely used for transmitting television (TV) signals occupying a spectrum in the VHF and UHF band of frequencies. In this form of amplitude modulation, one sideband is fully transmitted while a vestige or a part of the other sideband is transmitted. The carrier is also transmitted completely to aid the process of demodulation or picture signal recovery at the receiver through the use of simple envelope detectors.

**Television signals**

The exact details of modulation format used to transmit the video signal characterizing a TV system are influenced by two factors:

a) The video signal exhibits a large bandwidth and significant low frequency content, which rules out the possibility of using SSB. This is because SSB would require extremely extensive filtering to separate the two sidebands. In the presence of significant amount of low frequency contents which are necessary to reproduce the picture signal at the receiver, it is very difficult to suppress one sideband completely as the two sidebands are separated from each other by a small amount. Neither DSBSC is also useful as it requires a double bandwidth. Hence VSB becomes a choice that entails the transmission of one sideband completely and the other sideband being used partially.

b) The circuitry used for demodulation in the receiver should be simple and therefore cheap; this suggests the use of envelope detection, which requires the addition of a carrier to the VSB modulated wave.

With regard to point (a), it is to be noted that although there is indeed a basic desire to conserve bandwidth, in commercial TV broadcasting the transmitted signal is not quite VSB modulated. The reason is that at the transmitter power levels are high, with the result that it would be expensive to rigidly control the filtering of sidebands. Instead, a VSB filter is inserted in each receiver where the power levels are low. The overall performance is the same as conventional vestigial sideband modulation except for some wasted power and bandwidth.

**Sideband Shaping Filter in VSB**

Let us replace the sideband shaping filter by an equivalent complex lowpass filter of transfer function $\tilde{H}(f)$ as shown in Fig. The filter $\tilde{H}(f)$ may be expressed as the difference between two components $\tilde{H}_u(f)$ and $\tilde{H}_a(f)$ as

$$\tilde{H}(f) = \tilde{H}_u(f) - \tilde{H}_a(f)$$

The two components are described individually as follows:

a) The transfer function $\tilde{H}_u(f)$ as shown in Fig. pertains to a complex low pass filter equivalent to a bandpass filter designed to reject the lower sideband completely.
b) The transfer function $\tilde{H}_a(f)$ shown in Fig. accounts for both the generation of a vestige of the lower sideband and the removal of a corresponding portion from the upper sideband.

We may redefine the transfer function of the shaping filter as

$$\tilde{H}(f) = \begin{cases} \frac{1}{2}\left[1 + \operatorname{sgn}(f) - 2\tilde{H}_a(f)\right], & -f_a \le f < f_m \\ 0 & \text{elsewhere} \end{cases}$$

The signum function $\operatorname{sgn}(f)$ and the transfer function $\tilde{H}_a(f)$ are both odd functions of the frequency $f$. Hence they both have purely imaginary inverse Fourier transforms. Accordingly, we may introduce a new transfer function as

$$H_Q(f) = \frac{1}{j}\left[\operatorname{sgn}(f) - 2\tilde{H}_a(f)\right]$$

that has a purely real transfer function. Let $h_Q(t)$ denote the inverse Fourier transform of $H_Q(f)$; that is

$$h_Q(t) \Leftrightarrow H_Q(f)$$

Thus, our equivalent low pass shaping filter, in terms of the new filter becomes

$$\tilde{H}(f) = \begin{cases} \frac{1}{2}\left[1 + jH_Q(f)\right], & -f_a \le f < f_m \\ 0 & \text{elsewhere} \end{cases}$$

The VSB modulated signal is now derived in time domain. To do so, we write

$$s(t) = \operatorname{Re}\left[\tilde{s}(t)\exp(j2\pi f_c t)\right] \text{ (C)}$$

where $\tilde{s}(t)$ is the complex envelope of $s(t)$. Since $\tilde{s}(t)$ is the output of the complex low pass filter of transfer function $\tilde{H}(f)$ which is produced in response to the complex envelope of the DSBSC modulated signal, we may express the spectrum of $\tilde{s}(t)$ as

$$\tilde{S}(f) = \tilde{H}(f)\tilde{S}_{DSBSC}(f)$$

We have the complex DSBSC signal defined as

$$\tilde{S}_{DSBSC}(f) = E_c M(f)$$

Thus, the output of the equivalent shaping low pass filter is

$$\tilde{S}(f) = \frac{E_c}{2}\left[1 + j\tilde{H}_Q(f)\right]M(f)$$

Taking the inverse Fourier transform of the above we get

$$\tilde{s}(t) = \frac{E_c}{2}\left[m(t) + jm_Q(t)\right]$$

In the above, the quadrature component of the message signal $m_Q(t)$ is defined as

$$m_Q(t) = m(t) * h_Q(t)$$

Therefore, the VSB modulated signal becomes, from (C),

$$s(t) = \frac{E_c}{2}m(t)\cos 2\pi f_c t - = \frac{E_c}{2}m_Q(t)\sin 2\pi f_c t \quad \text{(D)}$$

As we observe, this is the desired representation of the VSB modulated signal containing a vestige of the lower sideband. The component $\frac{E_c}{2}m(t)$ is the in phase component of the modulated signal and the component $\frac{E_c}{2}m_Q(t)$ is the quadrature component.

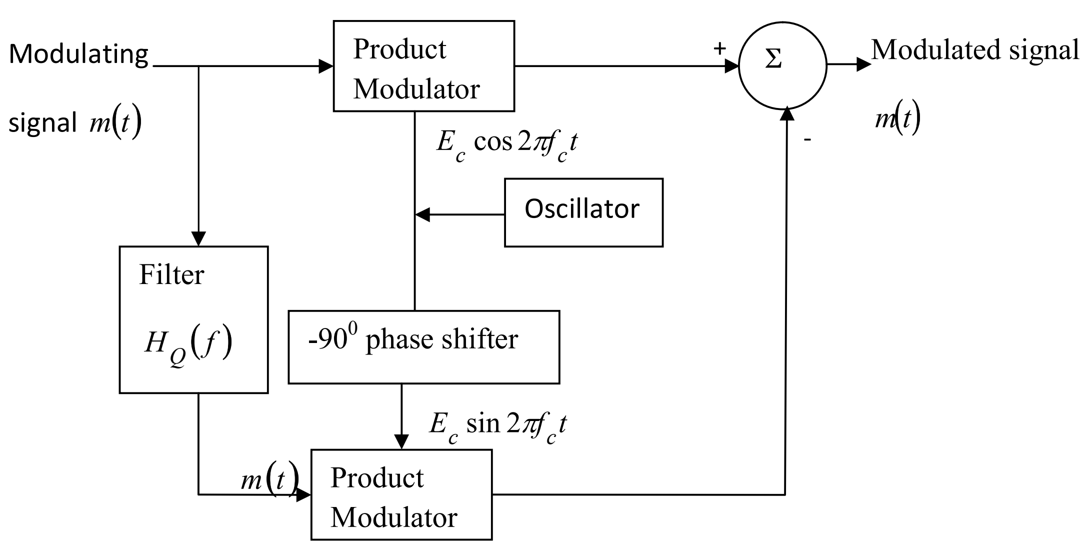


Fig. L 14.1 A method of generating VSB signal by a sideband shaping filter

The DSBSC and SSB signals may be considered to be two special cases of the VSB modulated signal as defined in (D). If the vestigial sideband is increased to the width of a full sideband, the resulting signal becomes a DSBSC wave with the result that $m_Q(t)$ vanishes. If, on the other hand, the width of the vestigial sideband is reduced to zero, the resulting signal becomes an SSB signal containing the upper sideband, with the result that $m_Q(t) = \hat{m}(t)$, where $\hat{m}(t)$ is the Hilbert transform of $m(t)$.

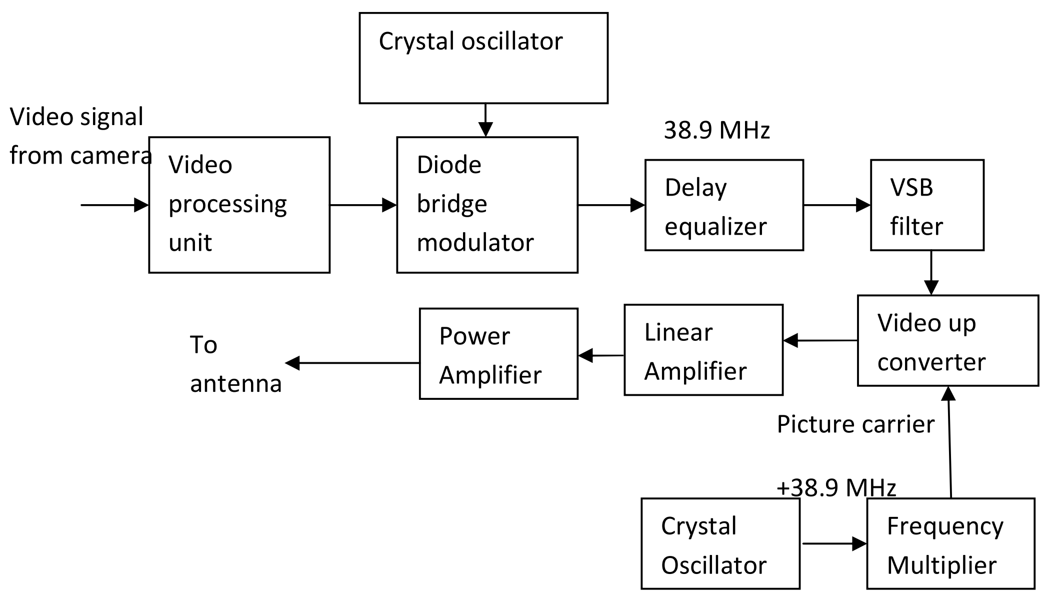



Fig. L 14.2 A portion of the TV transmitter to transmit picture signal only that uses low level modulation

A TV transmitter showing the use of VSB for transmission of video signals is illustrated in Fig. L 14.2.

## Quadrature Carrier Multiplexed System

This refers to the transmission of two independent baseband signals using the same carrier. The baseband signals DSBSC modulate a given carrier. As two independent message signals are transmitted simultaneously on the same carrier offset from each other in phase by $90^0$, it is known as quadrature carrier multiplexed modulation system. Both of the modulating signals require the same amount of bandwidth for transmission.

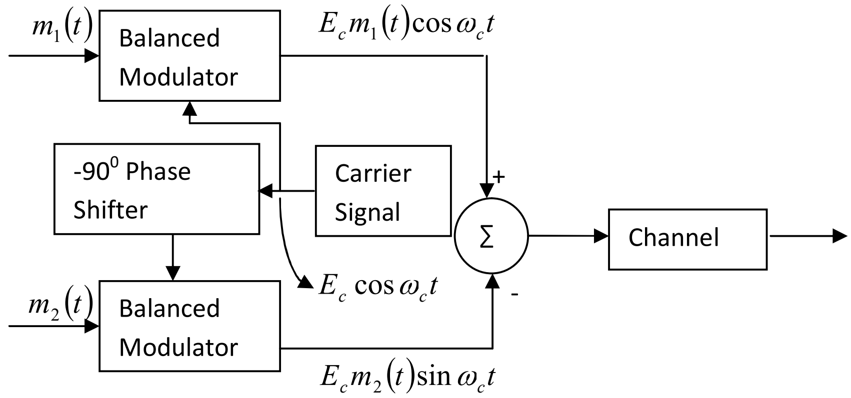



Fig. L 15.1 A transmitter that utilizes quadrature carrier multiplexing to transmit two independent message signals

The output at the summing junction is expressed as

$$y(t) = E_c\left[m_1(t)\cos\omega_c t + m_2(t)\sin\omega_c t\right]$$

Both of the modulated signals are DSBSC signals that require synchronous detection at the receiver. A block schematic of such a receiver is shown in Fig. L 15.2. In this diagram, we have not shown explicitly how frequency and phase synchronism is achieved between the transmitted and the regenerated carriers. However, this is also not important for our case now.

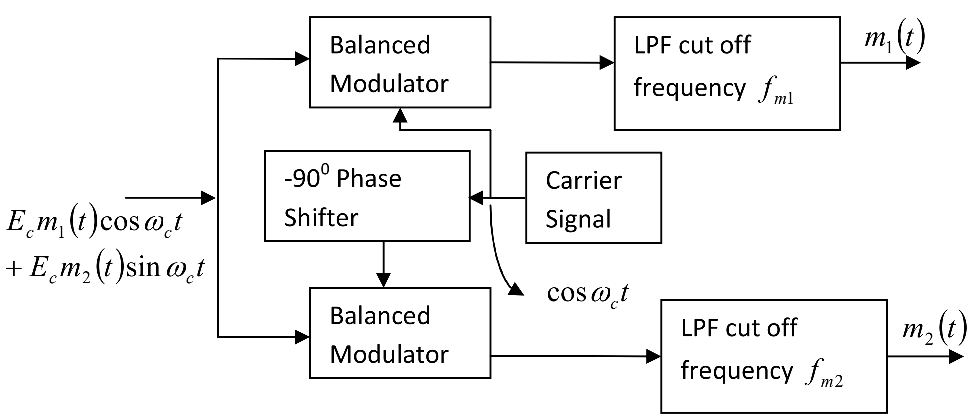



Fig. L 15.2 A receiver for detecting signals on quadrature carrier multiplexing

As we may observe quadrature carrier multiplexing reduces the requirement of number of subcarriers besides reducing the bandwidth required to transmit the multiplexed signal. A typical application of this scheme is used in color TV transmission wherein the color signals are transmitted simultaneously on two independent carriers.

# VESTIGIAL SIDEBAND MODULATION (VSB)

From previous lectures, it is to be noted that although there is indeed a basic desire to conserve bandwidth, in commercial TV broadcasting the transmitted signal is not quite VSB modulated. The reason is that at the transmitter power levels are high, with the result that it would be expensive to rigidly control the filtering of sidebands. Instead, a VSB filter is inserted in each receiver where the power levels are low. The overall performance is the same as conventional vestigial sideband modulation except for some wasted power and bandwidth.

## Filtering of Sidebands

Let the output from a product modulator be expressed as

$$u(t) = E_c m(t) \cos 2\pi f_c t$$

Let the transfer function of the bandpass filter following the product modulator be $H(f)$. Thus, the spectrum of the filtered modulated signal that appears at the output of the bandpass filter becomes

$$S(f) = U(f)H(f)$$
$$= \frac{E_c}{2}\left[M(f - f_c) + M(f + f_c)\right]H(f)$$

In the above, $M(f)$ denotes the spectrum of the message signal. The problem we address here is to design a filter transfer function  required to produce a modulated signal $s(t)$ with the desired spectral characteristics such that the original message signal may be recovered from $s(t)$ by coherent detection.

Coherent detection entails the multiplication of the incoming received signal with a locally generated carrier $E_c' \cos 2\pi f_c t$ that is synchronous with the transmitted carrier both in frequency and phase (let us ignore for the time being how this exact synchronism is achieved). Thus the receiver makes use of another product modulator whose output becomes

$$v(t) = s(t)E_c^{'} \cos 2\pi f_c t$$
$$S(f) = U(f)H(f)$$
$$= \frac{E_c}{2}\left[M(f - f_c) + M(f + f_c)\right]H(f)$$
$$= \frac{E_c}{2}M(f - f_c)H(f) + \frac{E_c}{2}M(f + f_c)H(f)$$
$$\Rightarrow S(f - f_c) = \frac{E_c}{2}M(f - 2f_c)H(f - f_c) + \frac{E_c}{2}M(f + f_c - f_c)H(f - f_c)$$
$$\Rightarrow S(f - f_c) = \frac{E_c}{2}H(f - f_c)\left[M(f - 2f_c) + M(f)\right]$$

and

$$\Rightarrow S(f + f_c) = \frac{E_c}{2}M(f + f_c + f_c)H(f + f_c) + \frac{E_c}{2}M(f + f_c - f_c)H(f + f_c)$$
$$\Rightarrow \frac{E_c}{2}H(f + f_c)\left[M(f + 2f_c) + M(f)\right]$$

Hence, we have

$$V(f) = \frac{E_c^{'}}{2}\left[S(f - f_c) + S(f + f_c)\right]$$
$$= \frac{E_c E_c^{'}}{4}M(f)\left[H(f - f_c) + H(f + f_c)\right]$$
$$+ \frac{E_c E_c^{'}}{4}\left[M(f - 2f_c)H(f - f_c) + M(f + 2f_c)H(f + f_c)\right]$$

The high frequency components of $v(t)$ represented by the second term are removed by a low pass filter that follows the product modulator. Thus, the spectrum of the low pass filtered signal output becomes

$$V_0(f) = \frac{E_c E_c^{'}}{4}M(f)\left[H(f - f_c) + H(f + f_c)\right]$$

For a distortionless reproduction of the original message signal at the coherent detector output, we require $V_0(f)$ to be a scaled version of $M(f)$ which further requires that

$$H(f - f_c) + H(f + f_c) = 2H(f_c) = \text{a constant}$$

We know that the message signal has a spectrum such that $M(f)$ is zero outside the interval of $-f_m \le f \le f_m$. Hence we need to satisfy the above equation for values of $f$ in this interval only. As a further simplification, we set

$H(f_c) = \frac{1}{2}$ so that

$$H(f - f_c) + H(f + f_c) = 1 \quad -f_m \le f \le f_m \quad \text{(A)}$$

We note that $s(t)$ is a bandpass signal. Hence, the canonical form of representation of it in terms of its inphase $s_I(t)$ and quadrature components $s_Q(t)$ become

$$s(t) = s_I(t)\cos 2\pi f_c t - s_Q(t)\sin 2\pi f_c t$$

We observe that, the spectrum of the inphase component is related to the modulated signal as

$$S_I(f) = \begin{cases} S(f - f_c) + S(f + f_c), & -f_m \le f \le f_m \\ 0 & \text{elsewhere} \end{cases}$$

This becomes

$$S_I(f) = U(f - f_c)H(f - f_c) + U(f + f_c)H(f + f_c)$$
$$= \frac{E_c}{2}M(f)\left[H(f - f_c) + H(f + f_c)\right], \qquad -f_m \le f \le f_m$$
$$= \frac{E_c}{2}M(f)$$

From the above, we get

$$s_I(t) = \frac{E_c}{2}m(t)$$

Now let us determine the quadrature component $s_Q(t)$. To do so, we first find out $S_Q(f)$ which is expressed as

$$S_Q(f) = \begin{cases} j\left[S(f - f_c) - S(f + f_c),\right] & -f_m \le f \le f_m \\ 0 & \text{elsewhere} \end{cases}$$

This takes the form of

$$S_Q(f) = j\left[U(f - f_c)H(f - f_c) - U(f + f_c)H(f + f_c)\right]$$
$$= j\frac{E_c}{2}M(f)\left[H(f - f_c) - H(f + f_c)\right], \qquad -f_m \le f \le f_m$$

A close look at the above expression tells us that the quadrature component $s_Q(t)$ can be generated from the message signal by passing it through a filter having a transfer function given as

$$H_Q(f) = j\left[H(f - f_c) - H(f + f_c)\right], \quad -f_m \le f \le f_m$$

Let $m'(t)$ denote the output of this filter in response to the message signal $m(t)$. Thus, the quadrature component of the modulated signal becomes

$$s_Q(t) = \frac{E_c}{2} m'(t)$$

Combining the inphase and the quadrature components of the modulated signal, we obtain

$$s(t) = \frac{E_c}{2} m(t)\cos 2\pi f_c t - \frac{E_c}{2} m'(t)\sin 2\pi f_c t \quad \text{(B)}$$

Two important points are made at this point:

a) The inphase component $s_I(t)$ is completely independent of the transfer function $H(f)$ of the bandpass filter involved in the generation of the modulated signal $s(t)$ so long as it satisfies (A).
b) The spectral modification attributed to the transfer function $H(f)$ is confined solely to the quadrature component $s_Q(t)$.

The role of the quadrature component is only to interfere with the inphase conmponent so as to reduce or eliminate power in one of the sidebands of the modulated signal , depending upon the application of interest.

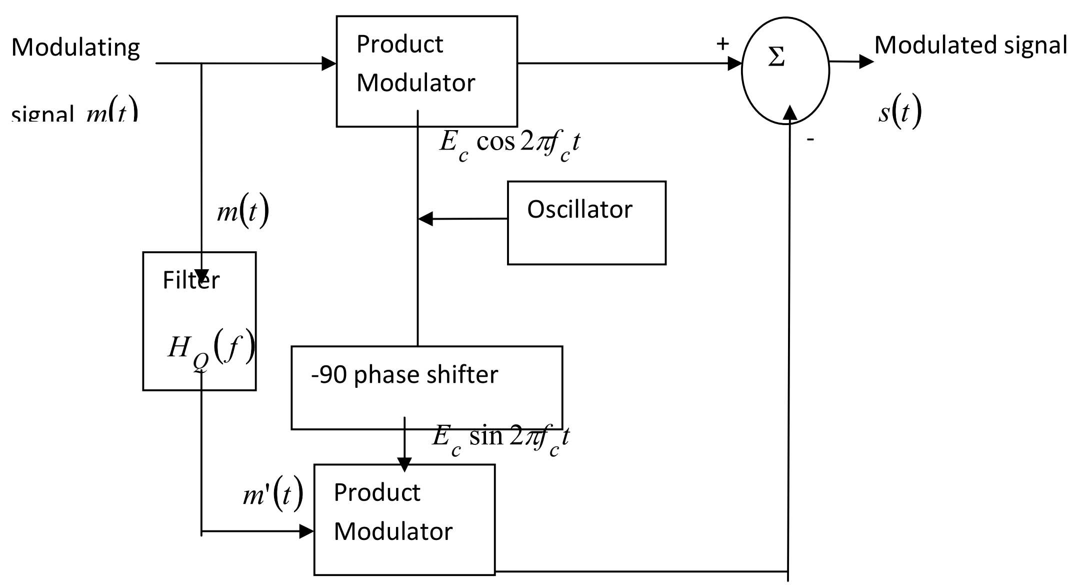



Fig. L 16.1 A schematic for implementing a VSB signal

**Envelope Detection of a VSB plus carrier**

Commercial analog television broadcasting makes use of VSB plus a sizeable amount of carrier to transmit video signal that occupies a bandwidth of typically 4-5 MHz. As it is a broadcast type of service, hence it represents a point-to-multipoint communication. Thousands of receivers need to be low cost which calls for envelope detection to be used in order to recover the video signal. It is therefore of interest to determine the distortion introduced because of envelope detector. The input to the envelope detector is

$$x(t) = A_c\left[1 + \frac{1}{2}k_a m(t)\right]\cos 2\pi f_c t - \frac{1}{2}k_a A_c m_Q(t)\sin 2\pi f_c t$$

Where $k_a$ is a constant that determines the percentage modulation. The output of the envelope detector is expressed as

$$a(t) = A_c\left\{\left[1 + \frac{1}{2}k_a m(t)\right]^2 + \left[\frac{1}{2}k_a m_Q(t)\right]^2\right\}^{\frac{1}{2}}$$

$$= A_c\left[1 + \frac{1}{2}k_a m(t)\right]\left\{\left[1 + \left[\frac{\frac{1}{2}k_a m_Q(t)}{1 + \frac{1}{2}k_a m(t)}\right]^2\right]\right\}^{\frac{1}{2}}$$

The rightmost term indicates the distortion contributed by the quadrature component $m_Q(t)$of the VSB signal. The distortion can be reduced by (i) reducing the percentage modulation to reduce $k_a$ and (ii) increasing the width of the vestigial sideband to reduce $m_Q(t)$. Both of these methods are used practically. In commercial TV broadcasting, the vestigial sideband occupies a width of about 1.25 MHz which amounts to about one-quarter of full sideband. This has been determined empirically as the width of the vestigial sideband modulation required to keep the distortion due to $m_Q(t)$ within tolerable limits when the percentage modulation is nearly 100.

## SUPERHETERDYNE RECEIVERS

The modulated signals, we learned in previous lectures are typically detected in radio receivers known as superheterodyne receivers. Edwin Armstrong invented the concept of super heterodyne receiver in 1918. A receiver is designed to carry out the inverse operation of a transmitter. Modulation is an important transmitter signal processing task that is decided by a host of factors such as the baseband signal type, the channel conditions, the simplicity and cost of a receiver and the type of application or service. Modulation of a carrier by a baseband signal is essentially a low pass to bandpass conversion that is effected by signal multiplication in time domain. Multiplication of a signal by a sinusoid shifts all frequencies up and down by the frequency of the sinusoid. Because of this, station selection can be accomplished by building a fixed bandpass filter and shifting the input frequencies so that the station of interest falls in the passband of the filter. This is analogous to constructing a viewing window on the frequency axis and instead of moving this window around to view a particular portion of the axis, we keep the window stationary and shift the entire axis. This shifting is called heterodyning and the resulting receiver is called a superheterodyne receiver. A typical receiver is shown in Fig. L 17.1.

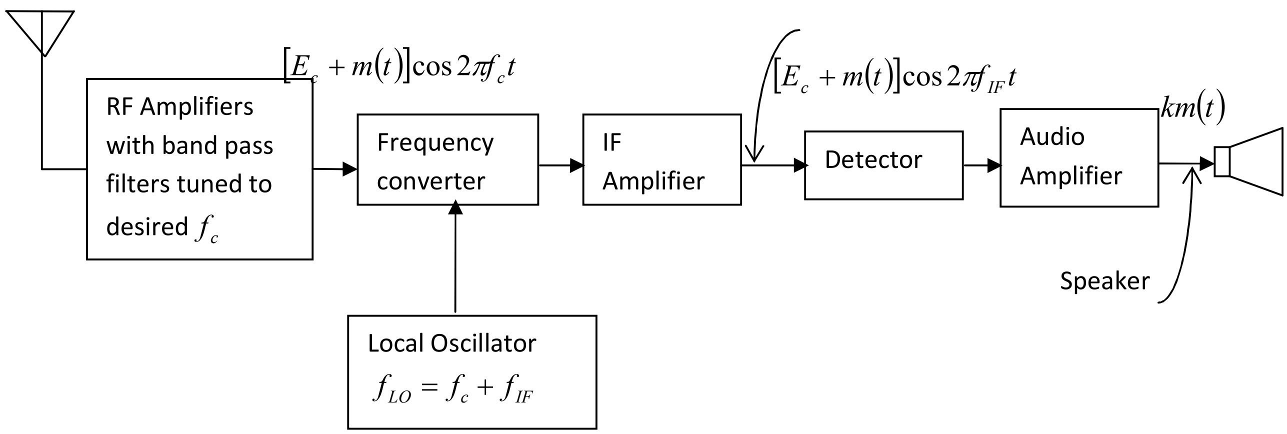



Fig. L 17.1 A superheterodyne radio receiver broadcast signals

Heterodyning produces both an upward and downward shift in frequency. While one of these shifts moves the desired station into the IF window (450 to 460 KHz), the other shift moves another station into this same window. This undesired signal is called an image and needs to be eliminated from the receiver.

A complete bandpass system consists of the transmission channel plus tuned amplifiers and coupling devices connected at each end. Hence, the overall frequency response has a more complicated shape than that of a single tuned amplifier. Various physical effects result in a loose but significant connection between the system's bandwidth and the carrier frequency $f_c$. The antennas in a radio system produce significant distortion unless the frequency range is small compared to $f_c$, moreover, design of a reasonably distortionless bandpass amplifier turns out to be quite difficult if the bandwidth $B$ is either very large or very small compared to $f_c$. As a rule of thumb, the fractional bandwidth $\frac{B}{f_c}$ should be kept within the range

$$0.01 < \frac{B}{f_c} < 0.1$$

The bandwidth of the system should be within 1% to 10% of the carrier frequency. Systems designed this way are called narrowband systems. All the communication systems that we see or work with fall into this category of narrowband systems unless otherwise mentioned.

As an example, let us listen to the Cuttack station operating at 972 KHz carrier frequency. The local oscillator is set to 972+455 = 1327 KHz. Multiplication by this sinusoid places the station at 972 KHz right into the IF filter passband. But the station operating at 1327+455 = 1782 KHz also multiplies the local oscillator frequency to produce a component at 455 KHz. This image station would be heard right on top of the desired station. The separation between the image and the desired station is twice the IF frequency or 910 KHz. A bandpass filter with a passband of less than 1820 KHz would accomplish the separation. This filter must pass the desired station, while rejecting the station 910 KHz away. This filter needs to be tunable also. But it need not be a sharp bandpass filter. A single stage of tuned circuit is adequate.

The antenna receives a signal that is a weighted sum of all broadcast signals. After some filtering to be examined later, the incoming signal is amplified in an RF amplifier. The resulting signal is shifted up and down in frequency by multiplying by a sinusoidal oscillator called the local oscillator. The output of the heterodyner is applied to sharp bandpass filter consisting of multiple filtering stages. This filtering is combined with amplification. The fixed band pass filter is set at 455 KHz, called the intermediate frequency (IF) and has a bandwidth of 10 KHz matching that of the station. In most receivers, the IF filter is made of three tuned circuits that are aligned so as to generate a Butterworth filter characteristics. The output of the IF amplifier represents a modulated signal with a fixed carrier frequency of 455 KHz with amplification and being separated out from the other signals.

Choice of the IF frequency

- Clearly, the IF frequency must not lie in the frequency band allotted for a given communication application. For example, commercial AM uses a frequency band from 535-1650 KHz. Thus, IF can not be taken to be any value inside this band.
- A very high value of IF would result in poor selectivity and poor adjacent-channel rejection unless sharp cutoff filters are used.

The incoming radio signal is given the advantage of image frequency rejection by the RF amplifier. All broadcast signals in the standard AM broadcast band (535-1605 kHz) are translated to a fixed frequency of 455 kHZ by the IF amplifier. The IF amplifier decides most of the gain and bandwidth of the radio receiver. This is in fact, an ingenious combination of amplifier and a bandpass filter. This is the greatest advantage of a superheterodyne receiver. After the signal is amplified, it is fed to an appropriate detector. This may be a noncoherent detector like an envelope detector that detects DSB with carrier type of signals or it may be any of the coherent detectors discussed previously. As we may observe, the original signal is obtained at the output of this detector which is power amplified (usually push pull configuration) and delivered to the speaker for reproducing the original speech or voice.

These receivers are usually equipped with automatic gain control (AGC) circuits that maintains the output from the speaker at a constant level in spite of variations occurring in the input signal. A part of the detector is tapped and given to IF amplifier input, mixer and the RF amplifier in a negative feedback manner so that if the demodulators output increases due to some reason, the IF amplifier is biased towards nearer the cutoff so that the gain reduces and vice versa otherwise.

## MODULE-III

## ANGLE MODULATION

Frequency modulation results when the frequency of the sinusoidal carrier is varied in accordance with the instantaneous changes in the amplitude of the modulating signal. The instantaneous frequency of the modulated signal becomes

$$\omega_i = \omega_c + k_f m(t)$$

This gives us the concept of instantaneous frequency; i.e. the frequency as a function of time as it is frequency of the carrier that keeps on changing in accordance with the modulating signal. If the modulating signal is analog, the frequency change is continuous. If the baseband signal is digital, then the frequency changes in a digital manner with respect to time. For example, if the modulating signal is a binary waveform that takes on only two amplitude levels, then the carrier frequency also changes in two steps; one frequency corresponding to a binary 0 and the other corresponding to a binary 1. These are also known as mark and space frequencies. This scheme is called binary frequency shift keying (BFSK). If we consider an m-ary waveform as the baseband signal, then the carrier frequency will also change in m-steps giving rise to a M-ary frequency shift keying (MFSK).

In a similar manner, if we write,

$$\phi_i = \phi + k_p m(t)$$

and substitute this as the phase of a sinusoidal carrier, then we obtain

$$v_{PM}(t) = V_c \sin(\phi_i) = V_c \sin(\phi + k_p m(t))$$

For a sinusoidal modulating signal, this takes on the form of

$$v_{PM}(t) = V_c \sin(\phi + k_p V_m \sin \omega_m t)$$

This is called phase modulation. Because the phase of the carrier is made to vary in accordance with the instantaneous value of the modulating signal. We note that, in adding the modulating signal to the phase, we have to take care of the dimensionality of the signal. This is so as we can add a phase component to another phase. Thus, the proportionality constant $k_p$ should be defined properly. For example, in case of a binary baseband signal, the phase is expected to change in two phases. One phase corresponds to binary zero and the other phase corresponding to binary one. Hence, we write,

$$v_{PM}(t) = V_c \sin(\phi + k_p.0) = V_c \sin\phi$$

for a binary zero. This means that the carrier is transmitted as such without any change in its amplitude, frequency and phase. However, for a binary one, we write,

$$v_{PM}(t) = V_c \sin(\phi + \pi.1) = -V_c \sin\phi$$

in keeping with the fact that we can add one phase to another. Hence $k_p$ assumes the value of $\pi$. The carrier is inverted in phase by 180 in correspondence with a binary one. This is called binary phase shift keying (BPSK). If a quaternary waveform is used as the modulating signal, the carrier phase change assumes four distinct values and this is called quaternary phase shift keying. In phase modulation, the instantaneous frequency is expressed as

$$\omega_i = \frac{d\phi_i}{dt} = k_p \frac{d}{dt} m(t)$$

which is proportional to the derivative of the modulating signal. For example, if the modulating signal is a sinusoid, the instantaneous frequency is proportional to a cosinusoid. A smooth time domain signal gives a continuous kind of instantaneous frequency. A square waveform, a trapezoidal waveform will yield abrupt changes in the instantaneous frequency. As we may note, the constant $k_p$ has the dimension of radians per volt. It is evident that, if we differentiate the modulating signal in the time domain and then this is used to frequency modulate a carrier, we obtain phase modulation. From the definition of a frequency modulated signal, we define what is called frequency deviation as

$$\Delta\omega = \omega_i - \omega_c = k_f m(t)$$

We thus find that, the instantaneous frequency deviation is directly proportional to the strength of the modulating signal. A signal with larger amplitude produces more frequency deviation and a signal with smaller amplitude gives rise to a lower frequency deviation. The maximum frequency deviation is, hence

$$\Delta\omega_{\max} = k_f \left|m(t)\right|_{\max}$$

The maximum frequency deviation is directly proportional to the peak amplitude of the signal. For a sinusoidal modulating signal, we have

$$\Delta\omega_{\max} = k_f V_m$$

or

$$\Delta f_{\max} = \frac{k_f V_m}{2\pi}$$

Thus,

$$\theta_i = \int \omega_i dt = \int \left(\omega_c + k_f m(t)\right)dt = \omega_c t + \int_{-\infty}^{t} m(\lambda)d\lambda$$

as $\lambda$ is a dummy variable.

Hence the expression for the modulated signal takes the following form

$$v_{FM}(t) = V_c \cos(\omega_i t + \phi) = V_c \cos\left(\omega_c t + k_f \int_{-\infty}^{t} m(\lambda)d\lambda + \phi\right)$$

For a sinusoidal signal $V_m \cos\omega_m t$, the frequency modulated signal looks like the following

$$v_{FM}(t) = V_c \cos(\omega_i t + \phi) = V_c \cos\left(\omega_c t + \frac{k_f V_m}{\omega_m}\sin\omega_m t + \phi\right)$$

Here we define the modulation index for an FM signal as

$$\beta = \frac{\Delta f_{\max}}{f_m} = \frac{k_f |m(t)|_{\max}}{f_m}$$

For all practical purposes, we may consider $\phi$ to be zero without any loss of generality.

From the expression of the FM signal, the modulation index $\beta$ is

$$\beta = \frac{k_f V_m}{2\pi f_m}$$

We note from the above that the modulation index for an FM signal is greater than unity unlike that in the DSB plus carrier type of AM. The modulation index here depends upon the peak amplitude of the modulating signal and its maximum frequency content. A standard value of the maximum frequency deviation is 75 KHz which is used for the FM broadcast systems. The FM broadcast systems operate in the 88-108 MHz. If we examine the expression for the FM signal, we note that it contains a term like cosine of a sine. The expansion of this term gives a cosine of another cosine and cosine of a sine. These are captured by Bessel's function. We write

$$v_{FM}(t) = V_c \cos(\omega_c t + \beta\sin\omega_m t) = V_c\left[\cos\omega_c t.\cos(\beta\sin\omega_m t) - \sin\omega_c t.\sin(\beta\sin\omega_m t)\right]$$

We write

$$\cos(\beta\sin\omega_m t) = J_0(\beta) + 2J_2(\beta)\sin 2\omega_m t + 2J_4(\beta)\sin 4\omega_m t + \ldots + 2J_{2n}(\beta)\sin 2n\omega_m t$$

$$\sin(\beta \sin \omega_m t) = J_1(\beta)\sin(\omega_m t) + 2J_3(\beta)\sin 3\omega_m t + 2J_5(\beta)\sin 5\omega_m t + ... + 2J_{2n-1}(\beta)\sin(2n-1)\omega_m t$$

Substitution of the two expressions in the expression for the FM signal gives us

$$v_{FM}(t) = V_c \begin{bmatrix} \cos\omega_c t\{J_0(\beta) + 2J_2(\beta)\sin 2\omega_m t + 2J_4(\beta)\sin 4\omega_m t + ... + 2J_{2n}(\beta)\sin 2n\omega_m t + ...\} \\ -\sin\omega_c t.\{2J_1(\beta)\sin(\omega_m t) + 2J_3(\beta)\sin 3\omega_m t + 2J_5(\beta)\sin 5\omega_m t + ... + 2J_{2n-1}(\beta)\sin(2n-1)\omega_m t ... +\} \end{bmatrix}$$

The bracketed term is simplified as

$$2\sin\omega_c t \sin\omega_m t = \cos(\omega_c + \omega_m)t - \cos(\omega_c + \omega_m)t$$

$$2\sin\omega_c t \sin n\omega_m t = \cos(\omega_c + n\omega_m)t - \cos(\omega_c + n\omega_m)t$$

$$2\cos\omega_c t \sin n\omega_m t = \sin(\omega_c + n\omega_m)t - \sin(\omega_c + n\omega_m)t$$

$$v_{FM}(t) = V_c \begin{bmatrix} J_0(\beta)\cos\omega_c t - J_1(\beta)\{\cos(\omega_c + \omega_m)t - \cos(\omega_c - \omega_m)t\} + J_2(\beta)\{\sin(\omega_c + 2\omega_m)t - \sin(\omega_c - 2\omega_m)t\} \\ -J_3(\beta)\{\cos(\omega_c + 3\omega_m)t - \cos(\omega_c - 3\omega_m)t\} + J_4(\beta)\{\sin(\omega_c + 4\omega_m)t - \sin(\omega_c - 4\omega_m)t\} - ... \end{bmatrix}$$

It is obvious from the above equation that, the FM signal contains a carrier term whose amplitude is $J_0(\beta)V_c$, two sidebands at frequencies $(\omega_c \pm \omega_m)$ with amplitude $J_1(\beta)V_c$, another pair of sidebands at frequencies $(\omega_c \pm 2\omega_m)$ with amplitude $J_2(\beta)V_c$ and so on. The sidebands occur at frequencies $(\omega_c \pm n\omega_m)$ with amplitude $J_n(\beta)V_c$. It is apparent that the spectrum of an FM signal extends up to infinity theoretically in both positive and negative frequency axes. And we may be led to the belief that the bandwidth of an FM signal is consequently infinite due to the presence of infinite number of sidebands. This is correct. Then do we require an infinite amount of bandwidth to transmit an FM signal? The answer is no. This is due to the fact that, although there are sidebands occurring at frequencies $(\omega_c \pm n\omega_m)$, however, their amplitudes vary as $J_n(\beta)V_c$ which assumes smaller values as $n$ becomes higher. Hence, it is practical to consider a few values of $n$ in the expression for the FM signal in order to compute its bandwidth. We may note, in passing that the spectrum of a phase modulated signal will look identical to that of the FM signal for the same value of modulation index.

Reproducing the expression for the FM signal, we note that, when the modulation index is very less, i.e. $\beta << 1$, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, then we call this a narrowband FM signal which has a simplified expression of

$$V_c \cos(\omega_c t + \beta \sin\omega_m t) = V_c[\cos\omega_c t.\cos(\beta\sin\omega_m t) - \sin\omega_c t.\sin(\beta\sin\omega_m t)]$$

and it becomes

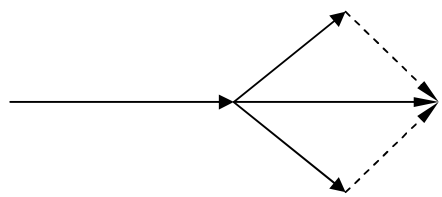
$$V_c \cos(\omega_c t + \beta \sin\omega_m t) = V_c[\cos\omega_c t - \sin\omega_c t.\beta\sin\omega_m t]$$

We expand this as

$$v_{NBFM}(t) = V_c\left[\cos\omega_c t - \frac{\beta}{2}\left(\cos(\omega_c - \omega_m)t - \cos(\omega_c + \omega_m)t\right)\right]$$

As we may note from the above expression, this looks similar to that obtained for a DSB+C type of AM waveform, however with certain differences. The AM waveform under consideration looks like

$$v_{AM}(t) = V_c\cos\left[\omega_c t + \frac{m_a}{2}\{\cos(\omega_c + \omega_m)t + \cos(\omega_c + \omega_m)t\}\right]$$



We observe from the above figure that, the resultant of the two sidebands in an AM waveform lies parallel to the phasor and points in the same direction as the carrier. Thus, the total amplitude at any instant is the sum of these two.

*Multiple Tone Wideband FM (Non Linear Modulation)*

We consider frequency modulation of a sinusoidal carrier when the modulating signal contains multiple sinusoids. For example, let us make

$$m(t) = V_{m1}\cos\omega_{m1}t + V_{m2}\cos\omega_{m2}t$$

This signal when frequency modulates a sinusoidal carrier gives

$$v_{FM}(t) = V_c\cos\left\lfloor\omega_c t + k_f\int m(t)dt\right\rfloor$$

Hence, $\int m(t)dt = \frac{V_{m1}}{\omega_{m1}}\sin\omega_{m1}t + \frac{V_{m2}}{\omega_{m2}}\sin\omega_{m2}t$

The resulting FM signal becomes

$$v_{FM}(t) = V_c\cos\left[\omega_c t + k_f\left\{\frac{V_{m1}}{\omega_{m1}}\sin\omega_{m1}t + \frac{V_{m2}}{\omega_{m2}}\sin\omega_{m2}t\right\}\right]$$

Let us denote

$$\beta_1 = \frac{k_f V_{m1}}{\omega_{m1}},\ \beta_2 = \frac{k_f V_{m2}}{\omega_{m2}}$$

The FM signal becomes in terms of the modulation indices,

$$v_{FM}(t) = V_c \cos[\omega_c t + \beta_1 \sin \omega_{m1} t + \beta_2 \sin \omega_{m2} t]$$

Expansion of this equation gives us

$$v_{FM}(t) = V_c [\cos \omega_c t.\cos\{\beta_1 \sin \omega_{m1} t + \beta_2 \sin \omega_{m2} t\}]$$

This further results in

$$v_{FM}(t) = V_c \begin{bmatrix} \cos \omega_c t.\{\cos(\beta_1 \sin \omega_{m1} t).\cos(\beta_2 \sin \omega_{m2} t) - \sin(\beta_1 \sin \omega_{m1} t).\sin(\beta_2 \sin \omega_{m2} t)\} \\ -\sin \omega_c t.\{\cos(\beta_1 \sin \omega_{m1} t).\cos(\beta_2 \sin \omega_{m2} t) - \sin(\beta_1 \sin \omega_{m1} t).\sin(\beta_2 \sin \omega_{m2} t)\} \end{bmatrix}$$

Let us expand the first curly bracketed term further and see what we get

$$\cos(\beta_1 \sin \omega_{m1} t).\cos(\beta_2 \sin \omega_{m2} t) = \sum_{m=0}^{2n} J_m(\beta_1) \sin m\omega_{m1} t.\sum_{p=0}^{2n} J_p(\beta_2) \sin m\omega_{m2} t$$

$$= \sum_{m=0}^{2n}\sum_{p=0}^{2n} J_m(\beta_1) J_p(\beta_2) \sin m\omega_{m1} t \sin p\omega_{m2} t$$

Use of $\sin(\beta \sin \omega_m t) = \sum_{l=0}^{2l+1} J_l(\beta) \sin(l\omega_m t)$ gives us

$$\sin(\beta_1 \sin \omega_{m1} t).\sin(\beta_2 \sin \omega_{m2} t) = \sum_{m=0}^{2m+1}\sum_{p=0}^{2p+1} J_m(\beta_1) J_p(\beta_2) \sin(m\omega_{m1} t) \sin(p\omega_{m2} t)$$

Similarly, the second curly bracketed term gives us

$$v_{FM}(t) = V_c \left[ \sum_{m=0}^{2n}\sum_{p=1}^{2n+1} J_m(\beta_1) J_p(\beta_2) \cos(\omega_c t + m\omega_{m1} t + p\omega_{m2} t) \right]$$

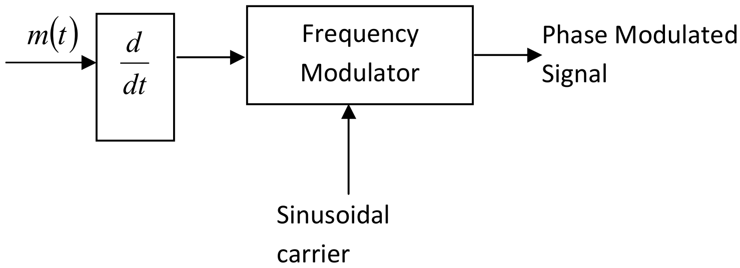
The expression contains the following terms

a) a carrier frequency $\omega_c t$ with an amplitude $J_0(\beta_1) J_0(\beta_2)$

b) a set of sidebands corresponding to the first sinusoid $\omega_{m1}$ with amplitudes $J_m(\beta_1) J_p(\beta_2)$ occurring at frequencies $(\omega_c + m\omega_{m1})$ for $m = 1,2,3,..$

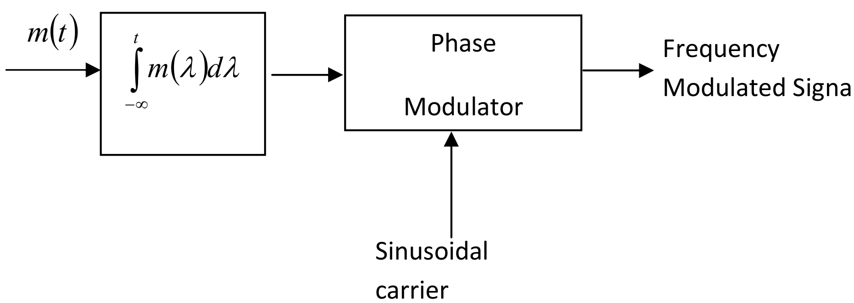
c) a set of sidebands corresponding to the first sinusoid $\omega_{m2}$ with amplitudes $J_m(\beta_1) J_p(\beta_2)$ occurring at frequencies $(\omega_c + p\omega_{m1})$ for $p = 1,2,3,..$

d) a set of cross modulation terms $(\omega_c + m\omega_{m1} + p\omega_{m2})$ with amplitudes $J_m(\beta_1) J_p(\beta_2)$ for $m = 1,2,3,..$ and $p = 1,2,3,..$




A phase modulator through the use of differentiation of the message signal

Similarly, we note that,




A frequency modulator through the use of integration of the message signal

These two figures illustrate the relationship between frequency modulation and phase modulation. If we integrate a signal in the time domain and then use this signal to phase modulate a sinusoidal carrier, we obtain a frequency modulated signal. Hence, this is called angle modulation and as we may note, is a nonlinear modulation system.

*FM as a nonlinear modulation system*

This follows from that fact that any sinusoid is expressed as

$$\mathrm{Re}\{V_c \exp(j\omega t)\}$$

For our case, the FM signal is

$$v_{FM}(t) = V_c \,\mathrm{Re}\left\{\exp\left[j\left(\omega_c t + k_f \int_{-\infty}^{t} m(\lambda)d\lambda\right)\right]\right\}$$

We find that,

$$\exp(x_1 + x_2) \neq \exp(x_1) + \exp(x_2)$$

where either $x_1$ or $x_2$ is taken as $j\left(\omega_c t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda\right)$.

This is in contrast with the DSB plus carrier type of AM system which is a linear modulation system.

POWER IN AN FM SIGNAL

We may find that, from the definition of the FM signal, the power in the FM signal is a constant. This is because, the power of a signal depends on its amplitude. The amplitude of an FM signal is a constant and hence the power is $V_c^2/2$ Watts. For a sinusoidal carrier of peak amplitude 1V, the power of the corresponding FM signal is 0.5 Watt. The information is contained in the frequency changes and the amplitude does not change. Hence power required to transmit an FM signal is a constant which is again in contrast with an AM signal. Any change in the amplitude of an FM signal is due to channel imperfections or distortions which are removed usually by means of a limiter in the receiver. It is due to this reason that an FM signal sounds clearer than an AM signal for the same program. The sound transmission in TV employs FM.

Comparison of AM and FM

Both FM and PM are called angle modulation and we have looked at circuits that can be used to transmit information. However, FM is preferred for practical systems. This is due to the fact that, in a PM waveform, information resides in the phase of the modulated carrier. In order to retrieve information from it at the receiver, we have to have perfect knowledge about phase. Maintaining a coherent phase for all possible values of phase in a PM waveform is an arduous task. This is because the phase of the carrier continually changes in response to an analog signal. Thus, system designers prefer to work with FM as extraction of the instantaneous frequency can be performed by a host of circuits. The FM has the following merits over AM.

1) The amplitude of the FM is a constant as information to be transmitted resides in the instantaneous frequency of the carrier. This amplitude is independent of the modulation index. Hence, the transmitted power in FM is a constant unlike AM where the modulation index decides the amount of power being transmitted. This implies that low level modulation may be performed in an FM transmitter facilitating the use of power efficient class C amplifiers in order to make the transmitted signal suitable for wireless transmission. All the transmitted power in FM is useful whereas in AM most of goes as a waste.
2) FM receivers are typically fitted with limiters that take care of amplitude fluctuation caused due to noise and thus, amplitude variations do not affect the quality of the reproduced signal. We say that an FM system enjoys more noise immunity.
3) It is possible to further reduce the effect of noise by increasing frequency deviation. AM does not have this feature as we cannot go for overmodulation in AM.
4) FM broadcast takes place in the upper VHF and UHF ranges which happen to be less noisy than the MF and HF allotted to AM broadcast.

5) FM broadcast takes place by space wave propagation limiting the radius of operation to slightly more than line of sight. It is possible to operate several independent transmitters on the same frequency with considerably less interference than would be possible with AM.

ANGLE MODULATOR CIRCUITS

In a varactor diode modulator, the junction capacitance of a reverse biased diode changes linearly with the modulating signal. The bias is varied by modulating voltage in series with a voltage of $-V_b$. This change in the junction capacitance of the diode brings about a change in the oscillating frequency of a suitable oscillator connected to this diode. This is the simplest reactance modulator and is often used for automatic frequency control and remote tuning.

Basic reactance modulator using FET

An FET based reactance modulator is shown in Fig. Here, the drain to gate impedance is assumed to be very large compared to the gate-to-source impedance. Three configurations of an FET based modulator are realized and shown in the following diagrams.

To determine $z$, a voltage $v_{ds}$ is applied between the drain and source. The resulting drain current $i_d$ is computed as follows. The gate voltage is

$$v_g = i_b R = \frac{v_{ds}}{R - jX_c} R$$

and the gate-to-source voltage is $v_{gs} = v_g - v_s = v_g$ as the source is grounded. This causes a drain current $i_d$ given as

$$i_d = g_m v_{gs} = \frac{g_m R}{R - jX_c} v_{ds}$$

The drain-to-source impedance thus, is

$$z = \frac{v_{ds}}{i_d} = \frac{v_{ds}}{\dfrac{g_m R v_{gs}}{R - jX_c}} = \frac{R - jX_c}{g_m R} = \frac{1}{g_m}\left(1 - \frac{jX_c}{R}\right)$$

As the reactance is much larger than the resistance, we will approximate the drain-to-source impedance as

$$z = -j\frac{X_c}{g_m R}$$

This means that the reactance is capacitive and we may write the drain-to-source impedance as

$$X_{eq} = \frac{X_c}{g_m R} = \frac{1}{2\pi f g_m RC} = \frac{1}{2\pi f C_{eq}}$$

The output impedance of the FET under these conditions is purely capacitive and is given as $C_{eq} = g_m RC$. Following observations are made from this equivalent capacitance.

- The expression $g_m RC$ has the dimension of capacitance
- The impedance $z$ would have a resistive component if the gate-to-source resistance is not small compared to the gate-to-drain impedance.
- The equivalent capacitance depends on the device transconductance and thus can be varied with respect to a voltage.
- The capacitance can be adjusted to any value , within limits, through judicious selection of $R$ and $C$.

In practice, the gate-to-drain impedance is made five to ten times the gate-source impedance. Let $X_c = nR$ at the carrier frequency. The, we have

$X_c = \dfrac{1}{\omega C} = nR$ and therefore $C = \dfrac{1}{\omega nR} = \dfrac{1}{2\pi f nR}$. Substitution of this value of capacitance into the equivalent capacitance obtained earlier, we get

$$C_{eq} = g_m RC = \frac{g_m R}{2\pi f nR} = \frac{g_m}{2\pi f n}$$

II. FET Reactance Modulator

We refer to Fig. where the places of $R$ and $C$ have been swapped and we further assume that the resistance is much larger than the reactance; i.e. $R >> X_c$. Everything else remaining the same, we write

$$v_g = i_b.\frac{1}{j\omega C} = \frac{v_{ds}}{R + \dfrac{1}{j\omega C}}.\frac{1}{j\omega C} = \frac{v_{ds}}{1 + j\omega RC}$$

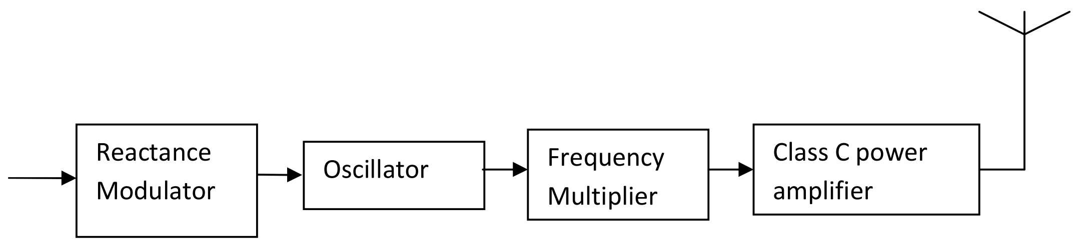
$$i_d = g_m v_g = \frac{g_m v_{ds}}{1 + j\omega RC}$$

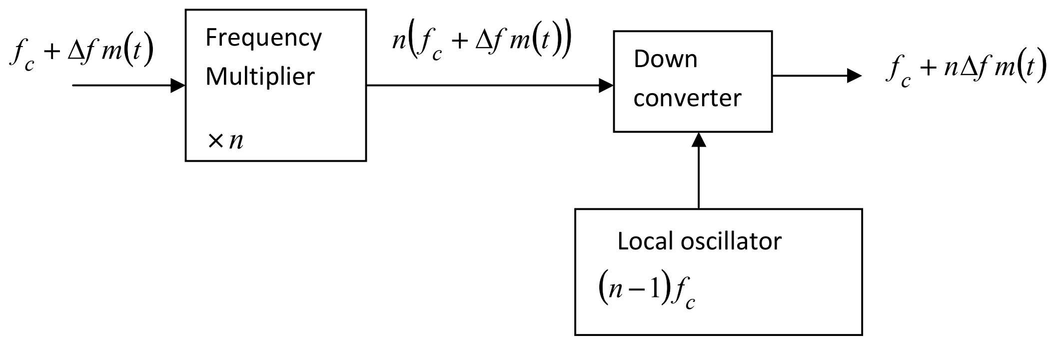
And therefore, the drain-to-source impedance is expressed as

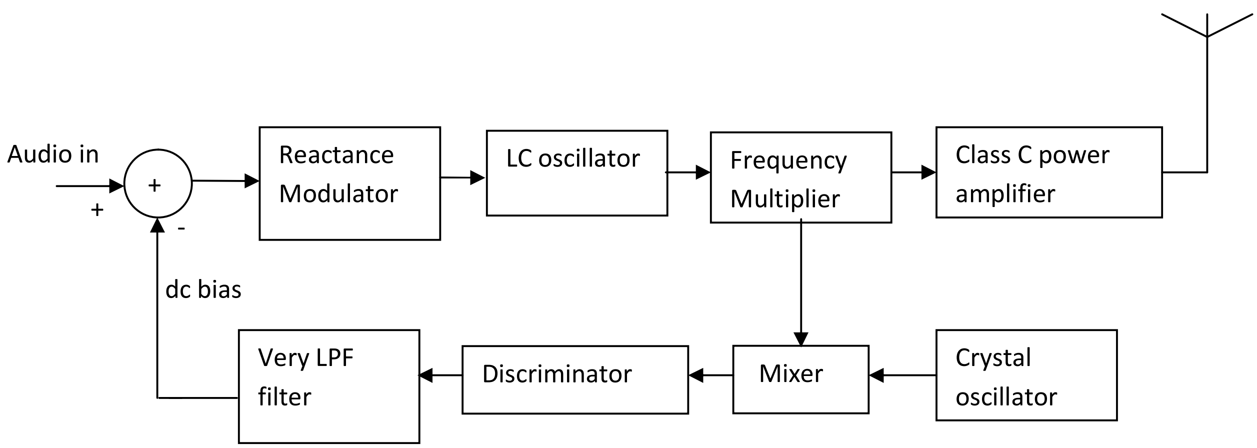
$$z = \frac{v_{ds}}{i_d} = \frac{1 + j\omega RC}{g_m} = \frac{R}{g_m}\left(\frac{1}{R} + j\omega C\right) \approx \frac{j\omega CR}{g_m}$$

The expression for $z$ shows that it is inductive, and the equivalent inductance is given as

$L_{eq} = \frac{RC}{g_m}$. The other two cases of FET reactance modulators are shown in the following figures.




A wide band FM transmitter

**FM Demodulators**

Frequency demodulation is the process that enables us to recover the original modulating signal from a frequency-modulated signal. The objective is to produce a transfer characteristic that is the inverse of that of the frequency modulator, which can be realized directly or indirectly. The requirement of a FM demodulator is to produce an output voltage that varies linearly with frequency. A direct method uses frequency discriminator that produces an instantaneous amplitude being proportional to the instantaneous frequency of the input FM signal. The slope detector is a very basic form of FM demodulator, though its linearity is not good.

The frequency discriminator is a combination of a slope circuit and an envelope detector. An ideal slope detector has an imaginary transfer characteristic that varies linearly with frequency inside a prescribed frequency interval.

Balanced Slope Detector

This is also called round Travis detector. It has two slope detectors, one tuned to a frequency above the carrier frequency while the other is tuned to below-the carrier frequency. The envelope detectors that follow the two slope detectors combine to produce a differential voltage. The output from this detector is observed to have an S shape when plotted as a function of frequency. When the incoming signal is unmodulated, the differential output voltage is then the incoming signal is unmodulated, the differential output voltage is zero as both the envelope detectors give identical outputs. When the carrier frequency is towards a higher frequency, one arm produces more voltage than the other and hence a positive voltage is obtained. On the other hand, when the carrier frequency deviates towards a lower frequency, it is the other arm that will produce more voltage than the other and hence the net differential output becomes negative.

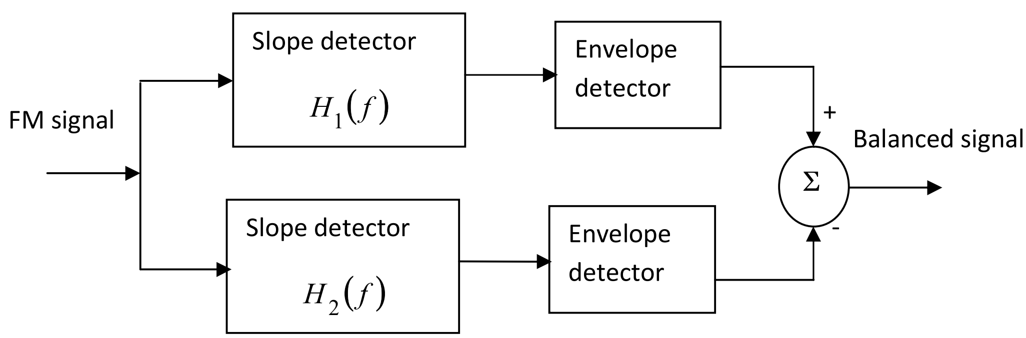



Fig. Block schematic of a frequency discriminator

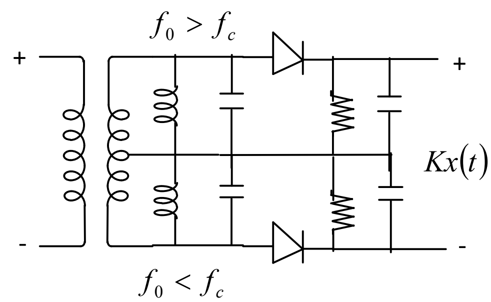
A simple frequency demodulator can also be realized by the following circuit.

Foster-Seeley Discriminator

This is also known as the center tuned discriminator. This is a derived form of the balanced slope detector and widely used in FM demodulators. Here both the primary and the secondary are tuned to the carrier frequency. This greatly simplifies the alignment problem of the balanced slope detector and yields better linearity. The voltage applied to each diode is the sum of the primary voltage and the corresponding half-secondary voltage. The primary and secondary voltages are:

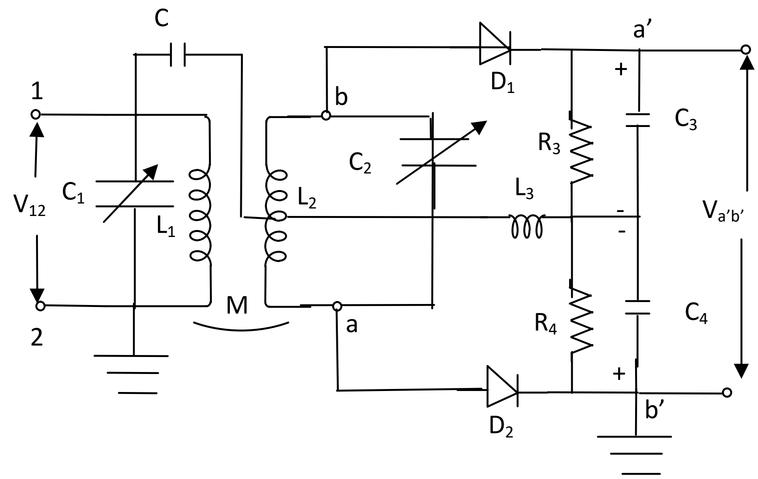
i) exactly $90^0$ out of phase for an input carrier frequency of $f_c$
ii) less than $90^0$ out of phase an input carrier frequency higher than $f_c$
iii) more than $90^0$ out of phase an input carrier frequency lower than $f_c$

This results in individual voltages being equal only for an incoming frequency equal to the carrier frequency. At all other values of carrier frequency, the output from one diode is higher than the other that depends on the deviation of the carrier frequency from its original value. The output magnitude depends on the deviation of the input frequency from the carrier frequency.




Phase Discriminator (Foster Seely Discriminator)

In this circuit, the individual component voltages will be the same at the diode inputs at all frequencies, the vector sums will differ with the phase difference between primary and secondary windings. The result is that the individual output voltages are equal only at the carrier frequency. At all other frequencies the output of one diode is greater than that of the other. Which diode has the larger output depends entirely on whether the incoming frequency is below or above the carrier frequency. It is noted that they are the same as in a balanced slope detector. Accordingly, the overall output is positive or negative according to the input frequency. As required, the magnitude of the output depends on the deviation of the input frequency from the carrier frequency.




Phase Discriminator circuit implementation for an FM demodulator

The resistances forming the load are made much larger than the capacitance reactances. The circuit composed of $C, L_3$ and $C_4$ is effectively placed across the primary winding. This is shown in Fig.2. The voltage across $L_3, V_L$ becomes

$$V_L = \frac{V_{12} Z_{L3}}{Z_c + Z_{C4} + Z_L} = V_{12} \frac{j\omega L_3}{j\omega L_3 - j\omega(1/\omega C + 1/\omega C_4)}$$

$L_3$ is an RF choke and is deliberately made large. Thus the inductive reactance greatly exceeds those of $C$ and $C_4$, especially since the first of these is a coupling capacitor and the second is an RF bypass capacitor. All of these imply that

$V_L \approx V_{12}$.Hence, it is proved that the voltage across the RF choke is equal to the applied primary voltage. The mutually coupled, double-tuned circuit has high values of primary and secondary $Q$ and a low mutual inductance. We may, therefore neglect the reflected resistance from the secondary and the primary resistance. The primary current is given as

$$I_P = \frac{V_{12}}{j\omega L_1}$$

The voltage induced in the secondary due to this primary current is

$$V_s = \pm j\omega M I_p$$

with the sign depending on the direction of winding. Let us work with the negative voltage. The secondary voltage becomes

$$V_s = -j\omega M I_p = -j\omega M \frac{V_{12}}{j\omega L_1} = -\frac{M}{L} V_{12}$$

The voltage across the secondary winding, $V_{ab}$, can now be calculated with the aid of Fig.3. The secondary has been redrawn here. It follows from this figure that

$$V_{ab} = V_s \frac{Z_{c2}}{Z_{c2} + Z_{L2} + R_2} = -\frac{jX_{C2}\left(-V_{12}M / L_1\right)}{R_2 + j\left(X_{L2} - X_{C2}\right)} = \frac{jM}{L_1} \frac{V_{12} X_{C2}}{R_2 + jX_{C2}}$$

where $X_2 = X_{L2} - X_{C2}$

and may be positive, negative or even zero, depending on the frequency. The total voltages applied to the two diodes may be written as

$$V_{ao} = V_{ac} + V_L = \frac{1}{2} V_{ab} + V_{12}$$

$$V_{bo} = V_{bc} + V_L = -V_{ac} + V_L = -\frac{1}{2} V_{ab} + V_{12}$$

The voltage applied to each diode is the sum of the primary voltage and the corresponding half-secondary voltage. The dc output voltages cannot be calculated exactly because the diode drop is not known. However, it is known that each is proportional to the peak value of the RF voltage applied to the respective diode. Hence,

$$V_{a'b'} = V_{a'o} - V_{b'o} = V_{ao} - V_{bo}$$

Let us consider the case when the input frequency $f_{in}$ is instantaneously equal to $f_c$. For this condition, $X_2$ is zero and the voltage becomes

$$V_{ab} = \frac{jM}{L_1} \frac{V_{12} X_{C2}}{R_2} = \frac{V_{12} X_{C2} M \angle 90^0}{R_2 L_1}$$

From the above equation, we note that, the secondary voltage $V_{ab}$ leads the applied primary voltage by $90^0$. Thus, $\frac{1}{2}V_{ab}$ leads $V_{12}$ by $90^0$, and $-\frac{1}{2}V_{ab}$ lags $V_{12}$ by $90^0$. Now we add up these two diode voltages vectorially. This is shown in the following figures. It is observed that, since $V_{ao} = V_{bo}$, the discriminator output is zero. For any incoming frequency other than the carrier frequency, there is a net output voltage. Let us consider the case when $f_{in}$ is less than $f_c$. Hence, $X_{L2}$ is less than $X_{C2}$ so that $X_2$ is negative. Hence, the output voltage becomes

$$V_{ab} = \frac{jM}{L_1} \frac{V_{12} X_{C2}}{R_2 + X_2} = \frac{V_{12} X_{C2} M \angle 90^0}{L_1 |Z_2| \angle -\theta^0} = \frac{V_{12} X_{C2} M}{L_1 |Z_2|} \angle (90 + \theta)^0$$

From this, it is observed that, $V_{ab}$ lags behind $V_{12}$ by more than $90^0$ so that $-\frac{1}{2}V_{ab}$ must lead $V_{12}$ by $90^0$. It is apparent from the vector diagram that $V_{ao}$ is less than $V_{bo}$. Thus, the discriminator output is negative when $f_{in}$ is less than $f_c$. Similarly, when the incoming frequency is greater than the carrier frequency, $X_2$ is positive, and the angle of the impedance $Z_2$ is also positive. Thus, $V_{ab}$ lags $V_{12}$ by less than $90^0$ . This time $V_{ao}$ is greater than $V_{bo}$ and the output voltage $V_{a'b'}$ is positive.

If the frequency response is plotted for the phase discriminator, it follows the required $S$

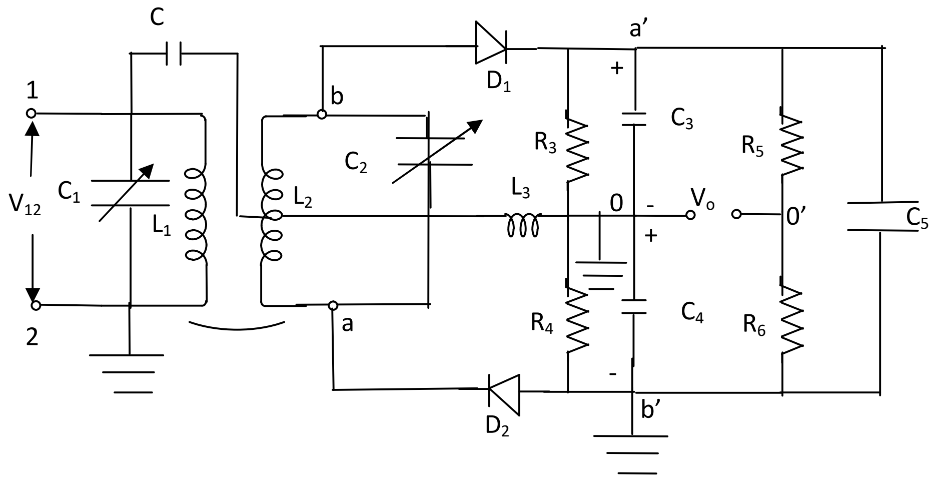
shape as shown in Fig.5. As the input frequency moves farther and farther away from the center frequency, the difference between the two diode input voltages becomes greater and greater. The output of the discriminator will increase up to the limits of the useful range, as shown in this figure. The limits correspond roughly to the half-power points of the discriminator tuned transformer. Beyond these points, the diode input voltages are reduced because of the frequency response of the transformer, so that the overall output falls.

The phase discriminator is much easier to align than the balanced slope detector. There are only tuned circuits, and both are tuned to the same frequency. Linearity is also better, because the circuit depends less on frequency response and more on the primary-secondary phase relation, which is quite linear. The only less noticeable disadvantage of this circuit is that it does not provide any amplitude limiting.

**Ratio Detector**

In the Foster-Seeley discriminator, changes in the magnitude of the input signal will give rise to amplitude changes in the resulting output voltage. This makes prior limiting necessary. A ratio detector addresses this problem by incorporating an amplitude limiter into the Foster-Seeley discriminator circuit.

A close look at the above FM detector reveals that the sum $V_{ao} + V_{bo}$ is a constant, although the difference keeps on changing with respect to the change in the incoming frequency. Deviation from this ideal does not result in undue distortion in the ratio detector. It follows that any variations in the magnitude of this sum voltage is considered undesirable. This needs to be suppressed. A discriminator that provides this suppression remains unaffected by the amplitude of the incoming signal. The ratio detector is obtained from the Foster-Seeley discriminator by i) reversing one diode, ii) placing a large capacitor $C_5$ across the output and iii) taking the output from elsewhere.




A ratio detector circuit

Reversing of the diode $D_2$ makes $o$ positive with respect to $b'$, so that $V_{a'b'}$ is now a sum voltage, rather than a difference voltage. Hence it becomes possible to connect a large capacitor between $a'$ and $b'$ in order to keep this voltage a constant. With the connection of this capacitor $C_5$, $V_{a'b'}$ does not represent the output voltage, rather the output voltage is taken between $o$ and $o'$. It is now necessary to ground one of these two points, and $o$ appears to be more convenient. In practice, $R_5 = R_6$, and hence the output voltage $V_o$ is calculated as

$$V_o = V_{b'o'} - V_{b'o} = \frac{V_{a'b'}}{2} - V_{b'o} = \frac{V_{a'o} + V_{b'o}}{2} - V_{b'o} = \frac{V_{a'o} - V_{b'o}}{2}$$

This equation shows that the ratio detector output voltage is equal to half the difference between the output voltages from individual diodes. Hence the output voltage is proportional to the difference between the individual output voltages. The ratio detector therefore behaves identically to the discriminator for input frequency changes. The $S$ curve applies equally to both the circuits.

The slope detectors-single or balanced- are not used in practice. They have been included here to gain an understanding of frequency-to-voltage conversion and help in building practical FM demodulators. The Foster-Seeley discriminator is very widely used in both narrowband and wideband FM radio receivers. It is also used in satellite station receivers, especially for the reception of TV carriers. The ratio detector is a good FM demodulator typically used in TV receivers for recovering frequency modulated audio signal. Its advantage over the discriminator is that it provides both limiting and a voltage suitable for AGC, while the main advantage of the discriminator is that it is very linear. Thus, the discriminator is preferred in situations in which linearity is an important characteristic (high-quality FM receivers), whereas the ratio detector is preferred in which linearity is not critical, but component and price savings. Under critical noise conditions as encountered in receiving satellite signals, the phase-locked loop is typically used.

Limiting of FM Waves

When an FM wave is transmitted through a communications channel, in general, the output is not expected to have a constant amplitude because of channel imperfections. At the receiver, it is essential to remove the amplitude fluctuations in the channel output prior to frequency demodulation. This is customarily done by means of an amplitude limiter. The transfer characteristic of an ideal hard limiter is shown in Fig.

We assume the limiter to be a memoryless device in order to analyze the operation of this circuit. The limiter output, in general can be expressed as

$$v(t) = \operatorname{sgn}[x_c(t)] = \begin{cases} +1 \ if \ x_c(t) > 0 \\ -1 \ if \ x_c(t) < 0 \end{cases}$$

We also assume the amplitude fluctuations to be slow compared to the zero-crossing rate of the FM wave $x_c(t)$. The sign changes of $x_c(t)$ may be considered to be proportional to the carrier phase shifts as given by

$$v(t) = \operatorname{sgn}\{\cos[\theta(t)]\}$$

where $\theta(t)$ is the phase component of the carrier containing the message signal. The function $\operatorname{sgn}\{\cos[\theta(t)]\}$ a function of $\theta$, is a periodic square wave when the modulation is zero. The Fourier series representation of this function gives us

$$\operatorname{sgn}\{\cos[\theta]\} = -\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{\cos[(2n-1)\theta]}{(2n-1)}$$

Use of $\theta(t)$ in place of $\theta$ in the above expression gives us

$$v(t) = -\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{\cos[(2n-1)[2\pi f_c t + \theta(t)]]}{(2n-1)}$$

From the above, we note that hard limiting the FM signal produces image sidebands at odd harmonics of the carrier frequency $f_c$. For a very large carrier frequency, a bandpass filter centered at $f_c$ selects the desired FM signal and rejects the higher order terms. The bandpass filter output, therefore becomes

$$v(t) = \frac{4}{\pi}\cos[2\pi f_c t + \theta(t)]$$

In practice, the combination of the hard limiter and band-pass filter is implemented as a single circuit commonly referred to as band-pass limiter.

A phase discriminator makes use of the following:

$$\cos(\omega t)\cos(\omega t+\theta)=\frac{1}{2}[\cos(2\omega t+\theta)+\cos\theta]$$

Use of a low pass filter with a cut off frequency of $\omega$ rad/s will eliminate the double frequency term and the output would be proportional to $\cos\theta$.

The composite signal at the IF filter output, is given as

$r(t)=v(t)+n(t)$ where $n(t)$ is a band-limited version of the white noise $w(t)$. In particular, $n(t)$ is the sample function of a noise process $N(t)$ with the following power spectral density:

PM Demodulators:

Phase modulators are the same as frequency modulators except that the signal is differentiated first and then fed to the VCO. We may approximate the process of differentiation by the following

$$x(t)-x(t-t_0)\approx t_0\frac{dx}{dt}$$

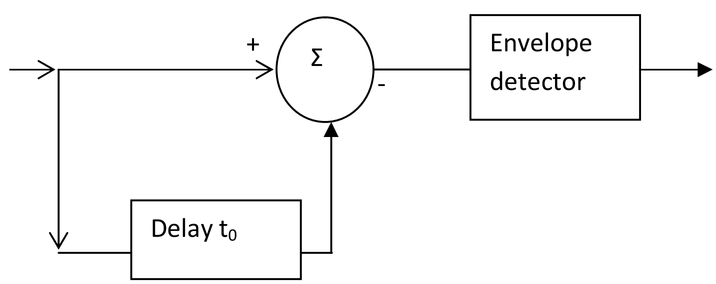



Fig. A phase demodulator

This leads to a demodulator as shown next. This is a phase demodulator as a time shift is equivalent to a phase shift. Any system that has a transfer function magnitude that is approximately linear with frequency in the range of frequencies of the FM wave changes FM into AM. Even a sloppy band pass filter will work as a discriminator if we operate over a limited range relative to the filter bandwidth, The linearity of a band pass filter discriminator can be improved by adopting the principles of a balanced modulator. The characteristic is subtracted from a shifted version of itself. The difference between the outputs of the two band pass filters with separate center frequencies is considered.

Let us assume that the modulated signal at the input to this circuit is

$$x_c(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$
$$= A_c \cos(2\pi f_c t).\cos[\beta \sin(2\pi f_m t)] - A_c \sin(2\pi f_c t).\sin[\beta \sin(2\pi f_m t)]$$

When the modulation index is less than unity and the delay produced is sufficiently small, we may approximate

$$\cos(2\pi f_m t_0) \approx 1 \text{ and } \sin(2\pi f_m t_0) \approx 2\pi f_m t_0$$

Under these conditions, the modulated signal is approximated as

$$x_c(t) == A_c \cos(2\pi f_c t).\cos[\beta(2\pi f_m t_0)] - A_c \sin(2\pi f_c t).\sin[\beta(2\pi f_m t_0)]$$
$$= A_c \cos(2\pi f_c t) - A_c \sin(2\pi f_c t).[\beta(2\pi f_m t_0)]$$

The delayed signal is

$$x_c(t - t_0) = A_c \cos[2\pi f_c (t - t_0) + \beta \sin(2\pi f_m ((t - t_0)))]$$

The IF used for TV is 40 MHz.

**MODULE-IV**
**Sampling of Analog Signals**

**4.1** Sampling of a band limited analog signal is of three types:

a) Instantaneous sampling
b) Flat top sampling and
c) Natural sampling

Instantaneous sampling is achieved by multiplying a band limited signal by a periodic impulse train. Let the periodic impulse train with period $T_s$ be represented as

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

The spectrum of this is

$$P(f) = f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s)$$

Multiplication of the message signal $m(t)$ and this periodic impulse train is identical to convolution in the frequency domain. Hence, we write

$$M_{samp}(f) = M(f) * P(f) = M(f) f_s \sum_{k=-\infty}^{\infty} \delta(f - kf_s) = f_s \sum_{k=-\infty}^{\infty} M(f - kf_s)$$

We observe that, after the multiplication, the resultant sampled signal becomes periodic with the amplitude of the impulse varying in proportion to the amplitude of the baseband signal. The spectrum is a line spectrum in the sense that the individual spectra are centered at integral multiples of the sampling frequency with a bandwidth equal to twice that of the original baseband signal. Hence, in this regard, an individual spectrum may be viewed as being equivalent to a DSBSC signal spectrum. Before we proceed further to understand the nuances of sampling, let us review a few concepts from fundamentals that we learned in module-I of this course.

Q 4.1: Show that $\int_{-\infty}^{\infty} \sin c(2\pi Bt - m\pi) \sin c(2\pi Bt - n\pi) dt = \begin{cases} 0 & m \neq n \\ \frac{1}{2B} & m = n \end{cases}$

Soln: The function can be written as

$$\sin c(2\pi Bt - k\pi) = \sin c\left[2\pi B\left(t - \frac{k}{2B}\right)\right] \Leftrightarrow \frac{1}{2B} rect\left(\frac{f}{2B}\right) e^{-jkf/2B}$$

The above integral is expressed as

$$\int_{-\infty}^{\infty}\frac{1}{2B}rect\left(\frac{f}{2B}\right)e^{-jmf/2B}\frac{1}{2B}rect\left(\frac{f}{2B}\right)e^{-jmf/2B}df=\left[\frac{1}{2B}rect\left(\frac{f}{2B}\right)\right]^2\int_{-B}^{B}\exp\left[\frac{j(m-n)f}{2B}\right]df$$

This result follows from the identity as shown below

$$\int_{-\infty}^{\infty}g_1(t)g_2(t)dt=\int_{\infty}^{\infty}G_1(f)G_2(-f)df=\int_{\infty}^{\infty}G_1(-f)G_2(f)df$$

We know that, $\int_{-\infty}^{\infty}\exp[j(n-m)]tdt=\begin{cases}1 & m=n\\ 0 & m\neq n\end{cases}$

Thus, the integral in our problem assumes a value equal to

$$\int_{-B}^{B}\exp\left[\frac{j(m-n)f}{2B}\right]df=\int_{-B}^{B}df=2B$$

Therefore,

$$\left[\frac{1}{2B}rect\left(\frac{f}{2B}\right)\right]^2\int_{-B}^{B}\exp\left[\frac{j(m-n)f}{2B}\right]df=\frac{1}{(2B)^2}.2B=\frac{1}{2B}\quad m=n$$

and zero for the case when $m\neq n$

**4.2** Reconstruction of the signal from its samples

A close look at the spectrum of the sampled signal reveals that it contains the original signal spectrum alongwith the other spectra centered at $\pm nf_s$. Hence to recover the original signal it suffices to pass the sampled spectrum through an ideal low pass filter or brick wall filter having the following frequency domain characteristics:

$$H(f)=\begin{cases}2f_m & |f|\leq f_m\\ 0 & \text{otherwise}\end{cases}$$

The output of this filter is

$$\hat{M}(f)=M_{samp}(f)H(f)$$

We observe from the above that, the rectangular filter passes only the baseband component having a maximum frequency content of $f_m$ Hz. Other spectral components are discarded at the output of the filter. Let us see the effect of rectangular filtering in the time domain.

Q 4.2: Assume that a bandlimited function, $s(t)=\dfrac{\sin 20\pi t}{\pi t}$ is sampled at 19 samples per second. The sampling function is a unit height pulse train with pulse widths of 1 msec. The sampled waveform forms the input to a low pass filter with cutoff frequency 10Hz. Find the output of the low pass filter and compare this with the original signal $s(t)$.

Soln: We only need to know the first two coefficients in the Fourier series expansion of the pulse train. These are given by

$$a_0 = \frac{0.001}{1/19} = 0.019$$

$$a_1 = 38 \int_{-5\times10^{-4}}^{5\times10^{4}} \cos 2\pi \times 19 t dt = \frac{2}{\pi} \sin\left(19\times10^{-3}\pi t\right)$$

The output time function is the inverse Fourier transform of $S_0(f)$, and is given by

$$s_0(t) = \frac{0.019 \sin 20\pi t}{\pi t} + 0.038 \frac{\sin \pi t}{\pi t} \cos 19\pi t$$

The second term represents the aliasing error.

Q 4.3: A 100Hz pulse train forms the input to the RC filter. The output of the filter is sampled at 700 samples per second. Find the aliasing error.

Soln: The square wave can be expanded in a Fourier series to yield,

$$v_{in}(t) = \frac{1}{2} + \frac{2}{\pi}\cos 2\pi\times100t - \frac{2}{3\pi}\cos 2\pi\times300t + \frac{2}{5\pi}\cos 2\pi\times500t...$$
$$= \frac{1}{2} + \sum_{n=1,nodd}^{\infty}(-1)^{\frac{n+3}{2}}\frac{2}{n\pi}\cos 2\pi n\times100t$$

The filter transfer function is given by

$$H(f) = \frac{1}{1+j2\pi fRC} = \frac{1}{1+j2\pi f(0.00167)}$$

The output of the filter is found by modifying each term in the input Fourier series. The amplitude is multiplied by the transfer function magnitude and the phase is shifted by the transfer function phase. The result is

$$v_0(t) = \frac{1}{2} + 0.45\cos\left(2\pi\times100t - 45^0\right) - 0.067\cos\left(2\pi\times300t - 71.6^0\right)$$
$$+ 0.025\cos\left(2\pi\times500t - 78.7^0\right) - 0.013\cos\left(2\pi\times700t - 81.9^0\right)$$

Let us assume impulse sampling. The result is that the component at 500Hz appears at 200Hz in the reconstructed waveform, and the component at 700Hz appears at dc (zero frequency). We shall ignore the higher harmonics. The reconstructed waveform is therefore given by

$$v_0(t) = \frac{1}{2} + 0.45\cos\left(2\pi \times 100t - 45^0\right) - 0.067\cos\left(2\pi \times 300t - 71.6^0\right)$$
$$+ 0.025\cos\left(2\pi \times 200t - 78.7^0\right) - 0.013\cos\left(-81.9^0\right)$$

The last two terms represent the aliasing error.

Q 4.4: The function $s(t) = \cos 2\pi t$ is sampled every ¾ second. Evaluate the aliasing error.

Soln: The impulse train of period $T_s$, each narrow impulse being of width $dt$ has a Fourier series expansion as

$$S(t) = \frac{dt}{T_s} + \frac{2dt}{T_s}\left(\cos 2\pi \frac{t}{T_s} + \cos 2 \times 2\pi \frac{t}{T_s} + \ldots\right)$$

The sampling period is $T_s = \frac{3}{4}$

Hence, $a_0 = \frac{0.001}{3/4} = \frac{0.004}{3} = 0.00133$ assuming $dt$ to be of 1 msec duration.

$$a_1 = \frac{0.002}{3/4}\cos 2\pi \times \frac{4}{3} t = \frac{0.008}{3}\cos\frac{8\pi t}{3} = 0.00266\cos(2.66t)$$

The aliasing error is due to $0.00266\cos(2.66t)$.

Q 4.5: A signal $m(t)$ is band-limited to $B$ Hz is sampled by a periodic pulse train $p_{T_S}(t)$ made up of a rectangular pulse of width $1/8B$ seconds (centered at the origin) repeating at the Nyquist rate (2B pulses per second). Show that the sampled signal $\overline{m(t)}$ is given by

$$\overline{m(t)} = \frac{1}{4} m(t) + \sum_{k=1}^{\infty} \frac{2}{k\pi} \sin\left(\frac{k\pi}{4}\right) m(t) \cos k\omega_S t \quad \omega_S = 4\pi B$$

Soln. The period of the periodic pulse train is $T_0 = \frac{1}{2B}$ as the pulses repeat at the rate of $2B$ pulses per second. The fundamental frequency is, therefore $f_0 = f_S = 2B$. The Fourier series of the periodic rectangular pulse train is written by computing the Fourier coefficients

$$a_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} m(t)dt = 2B \int_{-1/16B}^{1/16B} dt = 2B.\frac{1}{8B} = \frac{1}{4}$$

The coefficient $a_n$ is computed as

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} m(t)\cos\frac{2\pi nt}{T_0}\,dt = 4B\int_{-1/16B}^{1/16B}\cos\frac{2\pi nt}{T_0}\,dt = 4B.\frac{T_0}{2\pi n}\left|\sin\frac{2\pi nt}{T_0}\right|_{-1/16B}^{1/16B}$$

$$= 4B.\frac{1}{2B.2\pi n}\left\{\sin\frac{2\pi n(1/16B)}{T_0} - \sin\left[\left(-\frac{2\pi n(1/16B)}{T_0}\right)\right]\right\} = \frac{1}{\pi n}.2\sin\left(\frac{2\pi n(1/16B)}{T_0}\right)$$

We note that

$$\sin\left[\frac{2\pi n(1/16B)}{T_0}\right] = \sin\left(\frac{\pi n}{8BT_0}\right) = \sin\left(\frac{\pi n}{8B.1/2B}\right) = \sin\left(\frac{n\pi}{4}\right)$$

Therefore,

$$a_n = \frac{1}{\pi n}.2\sin\left(\frac{2\pi n(1/16B)}{T_0}\right) = \frac{2}{\pi n}\sin\left(\frac{n\pi}{4}\right)$$

Thus, the periodic rectangular pulse train is expressed as

$$m(t) = \frac{1}{4} + \sum_{n=1}^{\infty}\sin\frac{n\pi}{4}\cos n\omega_0 t = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{2}{\pi n}\sin\frac{n\pi}{4}\cos n\omega_S t$$

The sampled signal is obtained by simply multiplying the message signal by the periodic rectangular pulse train. We write,

$$\overline{m(t)} = m(t).p_{T_S}(t) = m(t).\left[\frac{1}{4} + \sum_{n=1}^{\infty}\frac{2}{\pi n}\sin\frac{n\pi}{4}\cos n\omega_S t\right]$$

$$= \frac{1}{4}m(t) + \sum_{n=1}^{\infty}\frac{2}{\pi n}\sin\frac{n\pi}{4}m(t)\cos n\omega_S t$$

$$\hat{m}(t) = m_{samp}(t) * h(t)$$

$$= \sum_{k=-\infty}^{\infty} m(kT_s)\delta(t - kT_s)$$

$$= m(t)\sum_{k=-\infty}^{\infty}\delta(t - kT_s) * h(t)$$

$$= m(t)\sum_{k=-\infty}^{\infty}\delta(t - kT_s) * \frac{2f_m}{2f_m}\sin c(2f_m t)$$

$$= \sum_{k=-\infty}^{\infty} m(t)\sin c[2f_m(t - kT_s)]$$

We observe from the above that the reconstructed signal $\hat{m}(t)$ is obtained by the superposition of sinc(x) pulses.

We observe from the above the following:

- The sampled signal contains a component proportional to the message signal $m(t)$.
- The signal has an envelope proportional to $\sin c\left(\frac{n}{4}\right)$
- The message signal can be recovered from the sampled signal by passing it through a low pass filter of cut off frequency $B$ Hz and gain 4.

The commutator approach toward multiplexing requires that the sampling rate of the various channels be identical. If signals with different sampling rates must be multiplexed, there are two general approaches that can be taken. One uses a buffer to store sample values and then intersperse these and spit them out at a fixed rate. The buffer approach is also effective if sampling rates contain variation (jitter). This is known as asynchronous multiplexing. The system must be designed so that the buffer always has samples to send when requested by the channel. This might require inserting stuffing samples if the buffer gets empty. Alternately, the buffer must be large enough so that it does not overflow with input samples.

The buffer approach is also used if the various sources are transmitting asynchronously. That is, suppose that they are not always transmitting information. The sizing of the buffer requires a probability analysis and the resulting multiplexer is known as a statistical multiplexer. The statistical multiplexer represents an efficient technique for multiplexing channels since a source only has a time slot when it needs it. On the negative side, since individual source messages are not occurring at a regular rate, the message must be tagged with a user ID. If the channels are synchronous with the samples occurring at a regular and continuous rate, the statistical multiplexer approach is not the best approach.

The second general technique involves sub-and super-commutation. This requires that all sampling rates be multiples of some basic rate. Meeting these requirements might require sampling some of the channels at a rate higher than what you would use without multiplexing. For example, if we have two channels with required sampling rates of 8KHz and 15.5KHz, in order to effect that combination we might choose to sample the higher frequency channel at 16KHz.

The concept of sub-and super-commutation is quite simple, and we illustrate it with an example. Let us suppose that we have a commutator wheel with 32 slots. Suppose we wish to multiplex the following 44 channels:

1 channel sampled at 80KHz

1 channel sampled at 40KHz

18 channels sampled at 10KHz

8 channels sampled at 1250Hz

16 channels sampled at 625 Hz

We note that all of the sampling rates are multiples of 625Hz. Let us choose the basic rate of the commutator wheel to be 10,000 rotations per second. Therefore, each of the 18 channels which must be sampled at 10KHz get one slot on the wheel. The channel that must be sampled at 40KHz needs four equally-spaced slots on the wheel, so it is sampled four times during each 0.1msec rotation of the wheel. Similarly, the 80KHz channel needs eight equally-spaced slots on the wheel. These higher rates are multiplexed using supercommutation.

The channels that need to be sampled at less than 10KHz only must be sampled on selected rotations of the wheel. For example, a 1250 channel needs to be sampled once every eight rotations of the wheel while a 625 Hz channel needs to be sampled only once every 16 rotations. We accomplish this using subcommutation wheels. The eight 1250Hz channels are commutated together with a wheel rotating at a rate of 1250 rotations per second. Each 0.1 msec, one of the channels is connected to a cell on the main commutator wheel. Similarly, the sixteen 625 Hz channels are commutated with a wheel rotating at 625 rotations per second.

Binary 1's and 0's such as in PCM signaling may be represented in various serial-bit signaling formats called line codes. Some of the widely used line codes are shown in the following figure. There are two major categories of the line codes: return-to-zero (RZ) and non-return-to zero (NRZ). With RZ coding, the waveform returns to a zero-volt level for a fraction (usually half) of the bit interval. Before discussing more about the line codes, let us touch upon some of the desirable aspects of a line code.

a) Self-synchronization- There is enough timing built into the code so that bit synchronizers can be designed to extract the timing or the clock signal. A long series of 1's and 0's should not cause a problem in timing recovery required at the receiver in order to establish the operating clock.

b) Low probability of bit error: Receivers can be designed that will recover the binary data with a low probability of bit error when the input data signal is corrupted by the noise or ISI.

c) Spectrum matching to the channel If the channel is ac coupled, the PSD of the line code should contain insignificant portions at frequencies near zero. In addition the signal bandwidth need to be sufficiently small compared to the channel bandwidth so that ISI will not be a serious issue.

d) Transmission bandwidth: It should be as small as possible.

e) Error detection capability: It should be possible to implement this feature easily by the addition of channel encoders and decoders, or it should be incorporated into the line code.

f) Transparency-The data protocol and line code are designed so that every possible sequence of data is faithfully and transparently received.

A quarternary signal may be formed by grouping the message bits in blocks of two and using four amplitude levels to represent the four possible combinations 00,01,10 and 11. Thus, $T = 2T_b$ and $r = \frac{r_b}{2}$. Different assignment rules or codes may relate $b_k$ to the grouped message bits. We show two such codes in Table 1.

Table No.1 Two codes for the line codes

| $b_k$ | Natural code | Gray code |
|---|---|---|
| $3A/2$ | 11 | 10 |
| $A/2$ | 10 | 11 |
| $-A/2$ | 01 | 01 |
| $-\frac{3A}{2}$ | 00 | 00 |

The Gray code has advantages relative to noise-induced errors because only one bit changes from going from level to level. Quaternary coding generalizes to $M-$ary coding in which blocks of $n$ message bits are represented by an $M$ -level waveform with

$$M = 2^n$$

Such a pulse corresponds to $n = \log_2 M$ bits. The M-ary signaling rate is decreased to

$$r = \frac{r_b}{\log_2 M}$$

We note that the use of $M-$ary coding reduces the requirement of transmission bandwidth by $\log_2 M$ as compared to binary transmission. However, increased signal power is required to maintain the same spacing between the amplitude levels. For an $M-$ary signaling format, the power associated with the signal is

$$\overline{b_k^2} = \left(\frac{A}{2}\right)^2 + \left(\frac{3A}{2}\right)^2 + \left(\frac{5A}{2}\right)^2 + ..\left(\frac{(2i-1)}{2}\right)^2 + \left(-\frac{A}{2}\right)^2 + \left(-\frac{3A}{2}\right)^2 + \left(-\frac{5A}{2}\right)^2 + ..\left(-\frac{(2i-1)}{2}\right)^2$$

This can be expressed as

$$\overline{b_k^2} = 2 \times \frac{1}{M}\sum_{i=1}^{M/2}(2i-1)^2\left(\frac{A}{2}\right)^2 = \frac{M^2-1}{12}A^2$$

*Design of a time-division multiplexer*

Q 4.6: Design a time-division multiplexer that will accommodate 11 channels. Assume that the sources have the following specifications.

Source 1. Analaog, 2-KHz bandwidth, Source 2. Analog, 4-KHz bandwidth, Source 3. Analog, 2-KHz bandwidth, Sources 4-11. Digital, synchronous at 7200 bits/sec

We also assume that the analog sources will be converted into 4-bit PCM words and, for simplicity, that frame sync will be provided via a separate channel and synchronous TDM lines are used. To satisfy the Nyquist rate for the analog sources, sources 1,2 and 3 need to be sampled at 4,8 and 4 KHz respectively. This can be accomplished by rotating the first commutator at $f_1 = 4$ KHz and sampling source 2 twice on each revolution. This produces a 16-kilosample/sec TDM PAM signal on the commutator output.Each of the analog sample values is converted into a 4-bit PCM word, so that the rate of the TDM PCM signal on the ADC output is 64 kbps. The digital data on the ADC output may be merged with the data from the digital sources by using a second commutator rotating at $f_2 = 8$ KHz and wired so that the 64-kbps PCM signal is present on 8 of the 16 terminals. This provides an effective sampling rate of 64kbps. On the other eight terminals the digital sources are connected to provide a data transfer rate of 8kbps for each source. Since the digital sources are supplying a 7.2-kbps data stream, pulse stuffing is used to raise the source rate to 8kbps.

The main advantage of TDM is that it can easily accommodate both analog and digital sources. However, when analog signals are converted to digital signals without redundancy reduction, they consume a great deal of digital system capacity.

Q 4.7: Consider a PCM TDM system in which 24 signals are to be processed. Each of the signals is bandlimited to 3.4 KHz and 8 bits are to be used for each quantized sample. Conventional NRZ-L encoding is used and an additional 8-bit sync word is placed in each frame. Find out the minimum bandwidth required.

Soln. The width of the shortest possible pulse needs to be determined in order to find out the bandwidth. The sampling rate is

2 x 3.4 = 6.8 KHz

The frame time is $\frac{1}{f_s} = \frac{1}{6.8} = 0.147 mS$ , The word time is $T_w = \frac{T_f}{k} = \frac{0.147}{25} = 0.00588 mS$

where the value of 24 represents the 24 data plus the sync word for each frame.

The bit interval is $\tau = \frac{0.00588}{8} = 0.000735 mS$

Hence the minimum transmission bandwidth is $\frac{0.5}{0.000735} = 680.272$ KHz

### 4.4 Digital PAM Signals

The PAM pulse train can be represented as

$$m(t) = \sum_k b_k p(t - kT) \qquad (1)$$

where the modulating amplitude $b_k$ represents the $k$ th symbol in the message sequence, so the amplitude belong to a set of $M$ discrete values. The index $k$ ranges from $-\infty$ to $\infty$ unless otherwise mentioned. The unmodulated pulse $p(t)$ may be rectangular or some other shape, subject to the condition

$$p(t) = \begin{cases} 1 & t = 0 \\ 0 & t = \pm T, \pm 2T, \pm 3T \ldots \end{cases} \qquad (2)$$

This condition is necessary to ensure the recovery of the message signal by sampling $m(t)$ periodically at $t = iT, i = 0, \pm 1, \pm 2, \ldots$ as

$$m(iT) = \sum_k b_k p(iT - kT) = b_k$$

The rectangular pulse $p(t) = \prod(t/\tau)$ satisfies the above equation (2) if $\tau \le T$ , as does any time limited pulse with $p(t) = 0$ for $|t| \ge \frac{T}{2}$. $T$ represents the pulse-to-pulse interval or the time allocated to one symbol. The signaling rate becomes $r = \frac{1}{T}$ measured in symbols per second or baud. In the binary case, the bit rate becomes

$$r_b = \frac{1}{T_b}$$

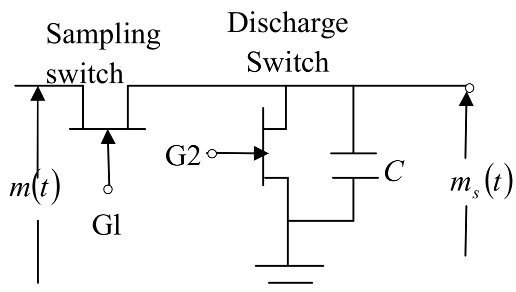



Fig 4.1. Flat Top sampling

In order to derive the power spectrum of the binary PAM waveform, under the assumption of independent and identically distributed (i.i.d) bits, we write

$$E[b_k b_j] = \begin{cases} \sigma_b^2 & j = k \\ 0 & j \neq k \end{cases}$$

As the rectangular pulse has a power spectrum of $T \operatorname{sinc}^2 fT$, the power spectrum of the binary PAM signal becomes

$$G(f) = \frac{\sigma_b^2}{T} |P(f)|^2$$

This is true if the PAM waveform has a mean value of zero. For a nonzero mean, the expression becomes

$$G(f) = \sigma_b^2 T \operatorname{sinc}^2 fT + m_b^2$$

where $m_b$ is its mean value. For unipolar signal formats, the ensemble average is given by the following autocorrelation function

$$R_b(n) = E[b_k b_{k-n}]$$

For a digital PAM signal having a pulse spectrum $P(f)$ and amplitude autocorrelation function $R_b(n)$, the power spectrum becomes

$$G(f) = \frac{1}{T} |P(f)|^2 \sum_{n=-\infty}^{\infty} R_b(n) e^{-j2\pi nfT}$$

For the case of uncorrelated symbols with a nonzero mean, we obtain

$$\sum_{n=-\infty}^{\infty} R_b(n) e^{-j2\pi nfT} = \sigma_b^2 + m_b^2 \sum_{n=-\infty}^{\infty} e^{-j2\pi nfT}$$

Use of Poisson's sum formula gives us

$$\sum_{n=-\infty}^{\infty} e^{-j2\pi nfT} = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$$

and hence we write

$$G(f) = \frac{\sigma_b^2}{T} |P(f)|^2 + \left(\frac{m_b}{T}\right)^2 \sum_{n=-\infty}^{\infty} \left| P\left(\frac{n}{T}\right) \right|^2 \delta\left(f - \frac{n}{T}\right)$$

This result shows that the power spectrum of a digital PAM signal has impulses at harmonics of the signaling rate $r$, unless the mean is zero or $P(f) = 0$ at all values of frequency. It is apparent from the above discussion that a synchronization signal can be obtained by applying $m(t)$ to a narrow BPF centered at one of these harmonic frequencies. The average power is obtained by integrating $G(f)$ over all $f$. Hence, $m^2$

For PTM Signals, under the assumption of uniform sampling, the duration of the $k$ th pulse is

$$\tau_k = \tau_0[1 + \mu m(kT_s)]$$

in which the unmodulated duration $\tau_0$ represents $m(kT_s) = 0$ and the modulation index $\mu$ controls the amount of duration modulation. The condition $1 + \mu m(kT_s)$ ensures no missing or negative pulses. The PPM pulses have fixed duration and amplitude and hence, unlike PAM and PWM, they do not suffer from the drawback of missing or negative pulses. The $k$ th pulse in a PPM signal begins at a time

$$t_k = kT_s + t_d + t_0 m(kT_s)$$

in which the unmodulated position $kT_s + t_d$ represents $m(kT_s = 0)$ and the constant $t_0$ controls the placement of the modulated pulse. Let us consider rectangular pulses with amplitude $A$ centered at $t = kT_s$ in order to have an informative approximation for the PWM waveform and let us further assume that $\tau_k$ varies slowly from pulse to pulse. Then the spectrum for these natural sampled waveform is

$$m_p(t) \approx Af_s\tau_0[1 + \mu m(t)] + \sum_{n=1}^{\infty} \frac{2A}{\pi n} \sin n\Phi(t) \cos n\omega_s t$$

where $\Phi(t) = \pi f_s \tau_0 [1 + \mu m(t)]$. From this equation, we observe that the PWM signal has a dc component in addition to the message signal and phase modulated waves at the harmonics of the sampling frequency $f_s$. The phase modulation has negligible overlap in the message band when $\tau_0 << T_s$ so that the message signal can be recovered by lowpass filtering with a DC block.

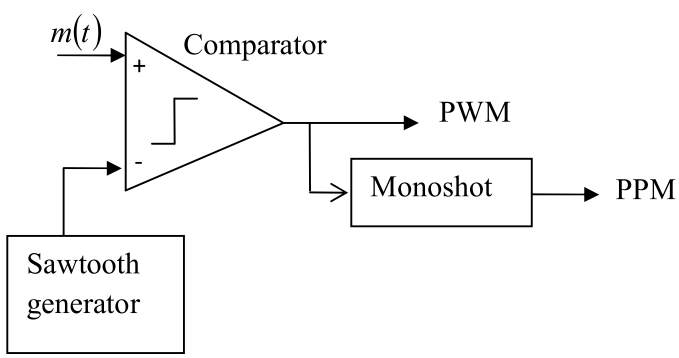



Fig 4.2. Generation of PWM and PPM

A very popular IC NE 555 has been used to generate PWM (PDM/PTM) and PPM signals is shown in Fig.4.3. (Students are encouraged to analyze the operation of this circuit and see how it generates a waveform whose width or duration is modified according to a message or modulating signal).

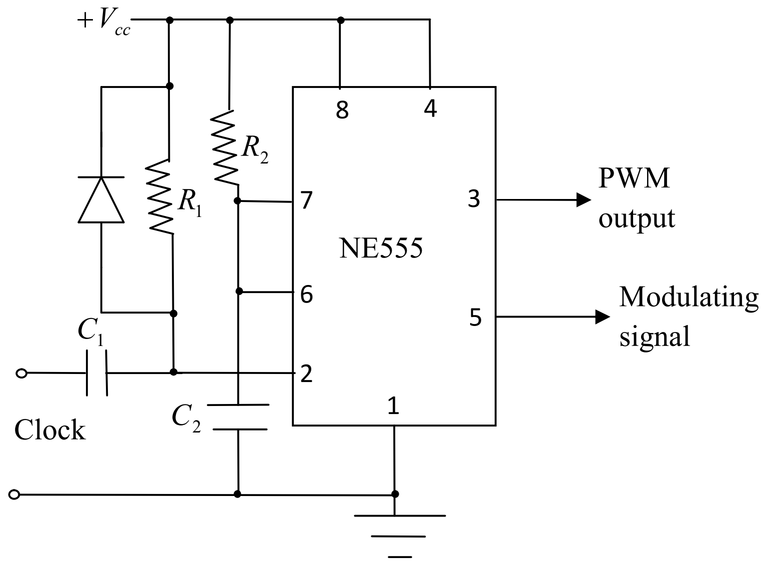



Fig. 4.3 An IC (NE 555 timer) based PWM modulator

Once we are able to generate a PWM signal, generation of a PPM signal is rather easier as it can be carried out by a differentiating circuit.

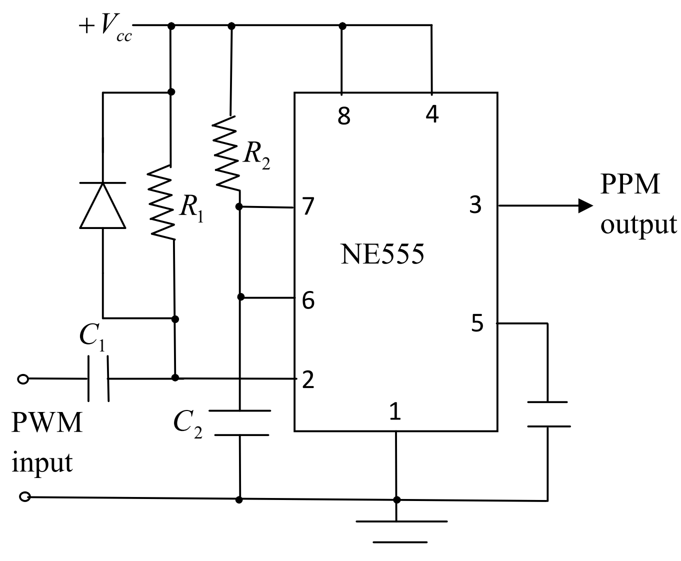



Fig. 4.4 PPM Modulator from PWM modulator

Message recovery in a PWM signal can also be carried out by converting the pulse-time modulation to pulse-amplitude modulation. To do so, we need to generate a ramp signal such as shown in Fig.4.5. This waveform is seen to start at time $kT_s$ and stops at $t_k$, restarts at $(k+1)T_s$. Both the start and stop epochs can be obtained from the edges of a PWM waveform whereas a PPM waveform must have an auxiliary synchronization signal for the start epoch.

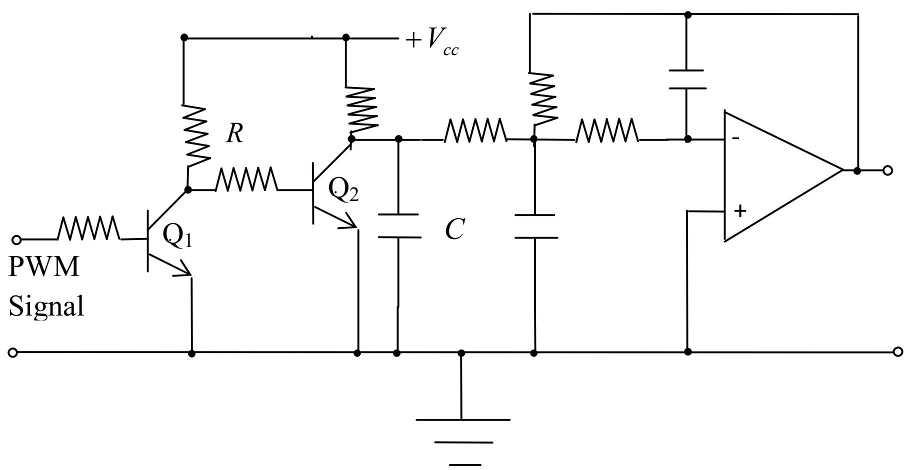



Fig. 4.5 A PWM Demodulator

The operation of the circuit is explained as follows. The BJT $Q_1$ acts as an inverter. The transistor $Q_2$ hence remains cut off during the high going portions of the incoming PWM signal. This allows the capacitor $C$ to get charged towards the biasing voltage through the resistor $R$. The time constant $RC$ is so chosen that before it can charge to $+V_{cc}$, the next pulse of the input signal arrives. If it is high, then $Q_2$ goes to the saturation condition which makes the capacitor discharge through the 'ON' transistor $Q_2$. The output at the collector of this transistor is, therefore a swatooth kind of waveform whose envelope follows the modulating signal. The second order low pass filter realized by the operational amplifier helps to recover the message or the modulating signal from this sawtooth waveform.

**MODULE-V**

**NOISE IN ANALOG MODULATED SYSTEMS**

## 5.1 Noise in AM receivers with envelope detection

The received signal at the envelope detector input consists of the modulated signal $m(t)$ at IF and the narrowband noise $n(t)$. This narrowband noise $n(t)$ is typically expressed in terms of its inphase component $n_I(t)$and the quadrature phase component $n_Q(t)$. Thus the received signal $r(t)$ becomes,

$$r(t) = m(t) + n(t) = A_c[1 + m_a m(t)]\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

$$= [A_c + A_c m_a m(t) + n_I(t)]\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

Thus, the envelope of the received signal becomes

$$y(t) = |r(t)| = \{[A_c + A_c m_a m(t) + n_I(t)]^2 + n_Q^2(t)\}^{1/2}$$

The signal $y(t)$ represents the output of an ideal envelope detector. The phase of the received signal is not of any interest to the envelope detector as it responds to the envelope of the received signal only and not to the phase changes. From the expression of the envelope, we note that it is the vector sum of two noisy components; one is the desired signal plus the inphase noise while the second term is noise only. In order to recover the original signal, we may immediately see that, the term $y(t)$ needs a simple manipulation as follows. Expansion of this term as a binomial expression and subsequent dropping of higher order terms give us

$$y(t) = |r(t)| \cong [A_c + A_c m_a m(t) + n_I(t)] + \frac{1}{2}[A_c + A_c m_a m(t) + n_I(t)]^{-1/2} n_Q^2(t) + \ldots + n_Q(t)$$

When the average carrier power is large compared with the average noise power, so that the receiver is operating satisfactorily, the signal term is usually larger than the noise terms $n_I(t)$ and $n_Q(t)$, most of the time. The second term in the RHS is usually very small compared to the first term and so the other terms following this in the series. Thus, the envelope of the received signal, is approximated as, to a good extent

$$y(t) = |r(t)| \cong A_c + A_c m_a m(t) + n_I(t)$$

The presence of the dc or the constant term $A_c$ in the envelope detector is due to the demodulation of the transmitted carrier. This term may be neglected as it does not contribute to the original message signal. This term may be removed simply by means of a blocking capacitor. We note that, the output of the envelope detector is the original signal, except for the scaling factor. Thus, the signal-to-noise ratio, at the envelope detector output is expressed as

$$(SNR)_{o,AM} \cong \frac{A_c^2 m_a^2 P}{2WN_0}$$

This expression is valid subject to the following conditions:

a) The noise at the receiver input is small compared to the carrier
b) The modulation index satisfies $m_a \le 1$

The figure of merit for an AM receiver is thus,

$$\frac{(SNR)_o}{(SNR)_C} \approx \frac{m_a^2 P}{1 + m_a^2 P}$$

**5.2 Noise in Angle Modulated Systems**

An angle modulated carrier, in general, is expressed as

$$x_c(t) = A_c \cos[\omega_c t + \phi(t)]$$

where $\phi(t) = k_p m(t)$ for PM and $\phi(t) = k_f \int_{-\infty}^{t} m(\lambda) d\lambda$ for FM

As we have noted from the discussion on noise in AM systems, the noise appearing at the output of the IF amplifier which is also the input to the demodulator is a bandpass noise with a PSD of $G_n(f)$ and bandwidth equal to that of the IF amplifier; i.e. $2(\Delta f + B)$.

This bandpass noise can be expressed in terms of the quadrature components as

$$n(t) = n_I(t)\cos \omega_c t + n_Q(t)\sin \omega_c t$$

where $n_I(t)$ and $n_Q(t)$ are both low pass signals of bandwidth $2(\Delta f + B)$. This noise can also be represented in terms of an envelope and phase as $n(t) = E_n(t)\cos[\omega_c t + \psi_n(t)]$.

Due to the nonlinear nature of angle modulation, superposition can not be applied. However, in special cases, the noise output is calculated by assuming the signal component to be zero. We derive the first the results for PM and extend these to the FM case.

**5.3 Phase Modulation**

The narrow-band modulation is assumed to be approximately linear. Therefore, we undertake the case of wideband frequency modulation. For such wide-band modulation, the signal changes very slowly compared to noise $n(t)$. The modulating signal bandwidth is $B$, and the noise bandwidth is $2(\Delta f + B)$ with $\Delta f >> B$. Thus, the phase and the frequency variations of the modulated carrier are much slower than are the variations of $n(t)$. The modulated carrier appears to have constant frequency and phase over several cycles, and hence, the carrier appears to be unmodulated. We, may therefore calculate the output noise by assuming the message signal to be zero or a constant. This is a qualitative argument for the linearity of wideband angle modulated signals. We outline a quantitative analysis as detailed below.

The demodulator input corresponding to phase modulation is given by

$$\begin{aligned} r(t) = m(t) + n(t) &= A_c \cos[\omega_c t + \phi(t)] + n(t) \\ &= A_c \cos[\omega_c t + \phi(t)] + E_n(t)\cos[\omega_c t + \psi_n(t)] \\ &= R(t)\cos[\omega_c t + \phi(t) + \Delta\phi(t)] \end{aligned}$$

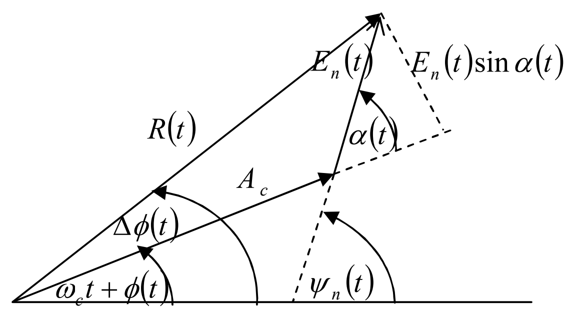



Fig. 5.1. Phasor diagram of the noisy FM signal appearing at the discriminator input

Expansion of this term gives us

$$r(t) = A_c[\cos\omega_c t.\cos\phi(t) - \sin\omega_c t.\sin\phi(t)] + E_n(t)[\cos\omega_c t.\cos\psi_n(t) - \sin\omega_c t.\sin\psi_n(t)]$$

Arranging the carrier terms of the above expression, we have

$$r(t) = [A_c \cos\phi(t) + E_n(t)\cos\psi_n(t)]\cos\omega_c t - [A_c \sin\phi(t) + E_n(t)\sin\psi_n(t)]\sin\omega_c t$$

Therefore,

$$R(t) = \left[A_c^2 + E_n^2 + 2A_c E_n \cos\{\phi(t) - \psi_n(t)\}\right]^{1/2},$$

$$\tan[\Delta\phi(t)] = -\frac{A_c \sin\phi(t) + E_n(t)\sin\psi_n(t)}{A_c \cos\phi(t) + E_n(t)\cos\psi_n(t)}$$

$\phi(t) = k_p m(t)$ for phase modulation. The resultant can also be written as

$$R(t) = A_c\left[1 + \frac{E_n^2}{A_c^2} + 2\frac{E_n}{A_c}\cos\{\phi(t) - \psi_n(t)\}\right]^{1/2}$$

We are interested in analyzing the effect of additive noise on the phase angle of the signal appearing at the discriminator input. The envelope term appearing in the above expression is not of any interest to us as information resides in the phase of the modulated signal. Hence, any change in the phase of the signal at the discriminator input due to noise is likely to bring about a change to the original signal. As this analysis is quite involved, we seek to simplify this by making reasonable assumptions about the SNR at the discriminator input. We consider the case of large SNR first. Under this situation, the phasor diagram corresponding to the actual phase, the extra phase shift introduced due to the additive noise is illustrated in Fig.

From Fig.5.1,

$$\sin[\Delta\phi(t)] = E_n(t)\sin\alpha(t)/R(t)$$

$$\text{where } \alpha(t) = \psi_n(t) - \phi(t)$$

For small noise case, $E_n(t) << A_c$ almost always, $\Delta\phi(t) << \frac{\pi}{2}$ for almost all $t$ and the resultant $R(t)$ is approximated as

$$R(t) \approx A_c$$

therefore,

$$\sin[\Delta\phi(t)] \approx \Delta\phi(t) = E_n(t)\sin\alpha(t)/R(t)$$

$$\Delta\phi(t) \approx \frac{E_n(t)}{A_c}\sin[\psi_n(t) - \phi(t)]$$

The discriminator detects the phase of the input and gives an output proportional to

$$x_0(t) = k_p m(t) + \frac{E_n}{A_c}\sin[\psi_n(t) - \phi(t)]$$

We observe from the above expression that, the noise has affected the phase of the modulated signal by adding one term to the original phase. As we have assumed the phase corresponding to the message signal to vary slowly than the $\psi_n(t)$ term, we approximate $\phi(t)$ by a constant $\phi$. Therefore,

$$\Delta\phi(t) \approx \frac{E_n}{A_c}\sin[\psi_n(t) - \phi] = \frac{E_n}{A_c}[\sin\psi_n(t).\cos\phi - \cos\psi_n(t).\sin\phi]$$

$$= \frac{1}{A_c}[n_Q(t)\cos\phi - n_I(t)\sin\phi]$$

The two quadrature components of white noise are uncorrelated to each other. Hence, the PSD corresponding to these two terms is

$$S_{\Delta\phi}(f) = \frac{\cos^2\phi}{A_c^2} S_{n_Q}(f) + \frac{\sin^2\phi}{A_c^2} S_{n_{I=}}(f) = \frac{S_{n_Q}(f)}{A_c^2}$$

This is because the PSDs corresponding to the two quadrature components are assumed to be equal.

For a white channel noise the PSDs are equal to $S_{\Delta\phi}(f) = \begin{cases} \dfrac{\eta}{A_c^2} & |f| \le \Delta f + B \\ 0 & otherwise \end{cases}$

The demodulated noise bandwidth is $\Delta f + B$. However, the useful signal bandwidth is only $B$ as the demodulated output passes through a low pass filter of cutoff frequency $B$ to remove the out of band noise. Thus, the PSD of the low pass filter output noise is

$$S_{n_0}(f) = \begin{cases} \dfrac{\eta}{A_c^2} & |f| \le B \\ 0 & otherwise \end{cases}$$

The output noise power, is, therefore,

$$N_0 = 2B.\left(\frac{\eta}{A_c^2}\right) = \frac{2\eta B}{A_c^2}$$

The signal power observed at the output of the demodulator is

$$S_0 = k_p \overline{m^2(t)}$$

The output SNR, of a PM receiver is therefore,

$$\frac{S_0}{N_0} = (A_c k_p)^2 \frac{\overline{m^2(t)}}{2\eta B}$$

These results are valid for small noise case and apply to both NBPM and WBPM. For PM, the maximum frequency deviation is expressed as

$\Delta f = k_p \dot{m}_p$ where $\dot{m}_p = \left[\dot{m}(t)\right]_{\max}$

Substitution of these in the output SNR gives us

$$\frac{S_0}{N_0} = \frac{(A_c \Delta f)^2}{2\eta B} \frac{\overline{m^2(t)}}{\dot{m}_p^2}$$

### 5.4 Noise in FM Systems

Frequency modulation can be viewed as a special kind of phase modulation, where the modulating signal is $\int_{-\infty}^{t} m(\lambda)d\lambda$ as illustrated in Fig. At the receiver, we demodulate FM with a PM demodulator followed by a differentiator. The PM demodulator output is $k_f \int_{-\infty}^{t} m(\lambda)d\lambda$.

The subsequent differentiator gives an output of the form $k_f m(t)$, so that we have, for the output signal power

$$S_0 = k_f^2 \overline{m(t)^2}$$

The phase demodulator output noise is identical to the one derived in the previous section with a PSD equal to $\frac{\eta}{A_c^2}$ for white channel noise. This noise is passed through an ideal differentiator that has a transfer function equal to $j2\pi f$. Hence the PSD of the output noise is $|j2\pi f|^2$ times the input PSD. We, therefore, write

$$S_{n0}(f) = \begin{cases} \frac{\eta}{A_c^2}(2\pi f)^2 & |f| \le B \\ 0 & otherwise \end{cases}$$

The output noise power is, hence,

$$N_0 = \int_{-B}^{B} \frac{\eta}{A_c^2}(2\pi f)^2 df = \frac{8\pi^2 \eta B^3}{3A_c^2}$$

Hence, the output SNR is

$$\frac{S_0}{N_0} = 3\left[\frac{k_f^2 \overline{m(t)^2}}{(2\pi B)^2}\right]\left(\frac{A_c^2/2}{\eta B}\right) = 3\left[\frac{k_f^2 \overline{m(t)^2}}{(2\pi B)^2}\right]\gamma$$

As $\Delta f = 2\pi k_f m_p$, we write

$$\frac{S_0}{N_0} = 3\left[\frac{k_f^2 \overline{m(t)^2}}{(2\pi B)^2}\right]\gamma = 3\left(\frac{\Delta f}{B}\right)^2\left(\frac{\overline{m(t)^2}}{m_p^2}\right)\gamma = 3\beta^2\left(\frac{\overline{m(t)^2}}{m_p^2}\right)\gamma$$

The transmission bandwidth is about $2\Delta f$. Hence, for each doubling of the bandwidth, the output SNR increases by 6 dB. Unlike PM, the output SNR does not increase indefinitely because of the appearance of threshold. This is because an increase in bandwidth results in a correspondingly increased noise power creeping into the system compared to the carrier power resulting in threshold.

For tone modulation,

$$\left(\frac{\overline{m(t)^2}}{m_p^2}\right) = 0.5$$

and thus, the output SNR becomes

$$\frac{S_0}{N_0} = \frac{3}{2}\beta^2\gamma$$

The output SNR in dB is plotted in Fig. as a function of $\gamma$ (also in dB) for various values of $\beta$. The dotted portion of the curve indicates the threshold region. Although the graphs in Fig. are valid for tone modulation only, they can be used for any other modulating signal simply by shifting them vertically by a factor of $\left(\frac{\overline{m(t)^2}}{m_p^2}\right)\Big/0.5 = 2\left(\frac{\overline{m(t)^2}}{m_p^2}\right)$. For tone modulation, we observe that FM is superior to PM by a factor of 3 dB. This does not mean that FM is superior to PM for other modulating signals as well. In fact, PM is better than FM for most practical signals. We write,

$$\frac{(S_0/N_0)_{PM}}{(S_0/N_0)_{FM}} = \left(\frac{(Bm_p)^2}{3m'^2_p}\right)$$

Thus, we observe that PM is better than FM from the output SNR point of view under the condition of $(Bm_p)^2 > 3m'^2_p$. If the PSD of the message signal is concentrated at lower frequencies, low frequency components predominate it and $m'_p$ is small. This favors PM. Thus, in general, PM is better than FM for message signals having predominant low frequency components (like the video signal) and FM is better than PM for message signals that have an abundance of high frequency components. This explains the better SNR of FM than PM for tone modulation as all the signal power is concentrated in the highest frequency band. But for most of the practical signals, the signal power is usually concentrated at lower frequencies and this makes PM a better candidate than FM for the modulation choice.